Spectroscopy by Tensor Renormalization group method

~ how to obtain energy spectrum and determine quantum number using tensor networks with Lagrangian formalism

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- Spectroscopy for lattice QCD with Monte Carlo method
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  - How to determine quantum number
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Spectroscopy for lattice QCD with Monte Carlo method

Energy spectrum

- Schrödinger eq.

(e.g., 0^{-+} for pion)

quantum number : $J^{PC}, \text{flavors}, \cdots$

$$\hat{H}|n,q\rangle = E_{n,q}|n,q\rangle$$

QCD Hamiltonian

$$(n = 0, 1, 2, \cdots)$$

$\hat{H}|\Omega\rangle = 0$: vacuum

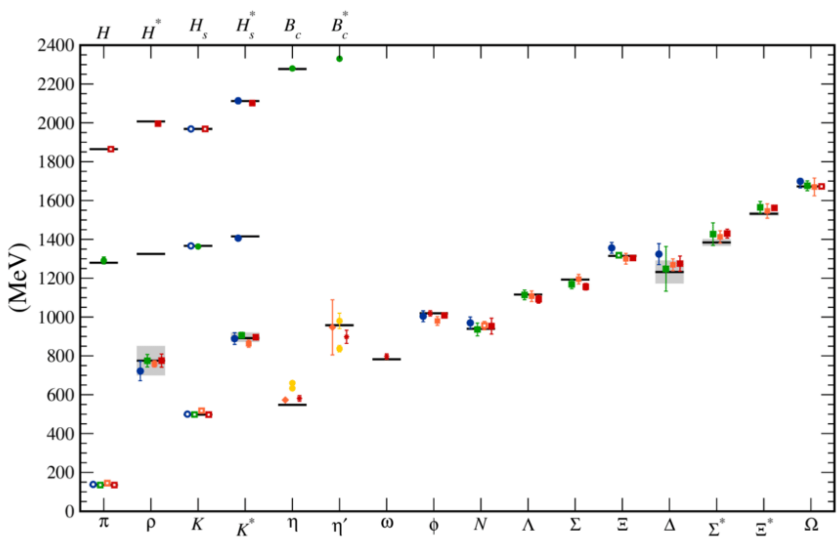
- Two-point function (Euclidean time)

$$\lim_{\beta\to\infty} \text{Tr}\left[\hat{\mathcal{O}}_q^\dagger(\tau)\hat{\mathcal{O}}_q(0)e^{-\beta\hat{H}}\right] = \sum_{n=0}^{\infty} |\langle\Omega|\hat{\mathcal{O}}_q(0)|n,q\rangle|^2 e^{-\tau E_{n,q}}$$

$$\hat{1} = \sum_{n,q'} |n,q'\rangle\langle n,q'|$$

Hadron spectroscopy

$$\lim_{\beta\to\infty} \mathrm{Tr}\left[\hat{\mathcal{O}}_q^\dagger(\tau)\hat{\mathcal{O}}_q(0)e^{-\beta\hat{H}}\right] = \sum_{n=0}^{\infty} |\langle\Omega|\hat{\mathcal{O}}_q(0)|n,q\rangle|^2 e^{-\tau E_{n,q}}$$



2013 snowmass report

Hadron spectroscopy

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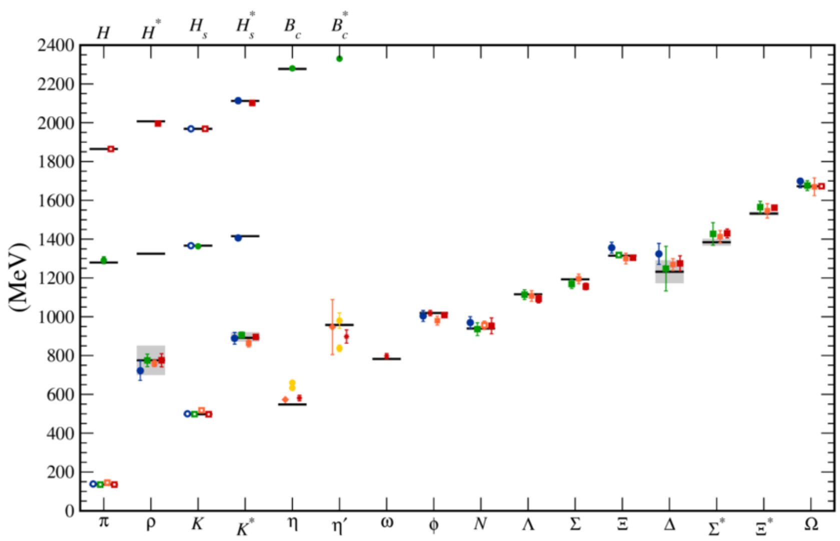

2013 snowmass report

Problems:

- Need large time extent β and time separation τ
- Need large statistics to extract higher excited states

How to get spectrum by TN?

- Hamiltonian formalism

  ⇒ Matsumoto's talk arXiv:2307.16655

- Lagrangian formalism
  - Two-point function
    - Large time extent and separation are easily realized
    - nothing new! (just do it)
  - Transfer matrix We use here!
    - No need to extrapolate time extent and separation

Transfer matrix formalism

TM and its spectrum

$$Z = \mathrm{tr}\ [\mathcal{T}\mathcal{T}\cdots\mathcal{T}]$$

for 2D Ising

$s_x = \pm 1$

$$\mathcal{T}_{s's} = \left(\prod_x e^{\beta s'_x s_x}\right)\left(\prod_x e^{\frac{1}{2}\beta s'_{x+1}s'_x + \frac{1}{2}\beta s_{x+1}s_x}\right)$$

: Hermitian, (semi)positive definite

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: Hermitian, (semi)positive definite

$$\Rightarrow \quad \mathcal{T} = Ue^{-\omega}U^\dagger \qquad \left(\mathcal{T}|a\rangle = e^{-\omega_a}|a\rangle \ \text{ for } \ a = 0, 1, 2\cdots\right)$$

$$\omega_a \geq 0 \qquad \omega_0 = 0$$

$$\mathcal{T} \leftrightarrow e^{-\hat{H}} \quad \Rightarrow \quad \omega_a \leftrightarrow E_{n,q}$$

How to identify quantum number of $|a\rangle$?

How to identify quantum number

Answer: compute matrix elements $\langle b|\hat{\mathcal{O}}_q|a\rangle$

eigenstates of TM

q is assumed to be well known

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Why:

For conserved charge of continuous symmetry $\hat{Q}$ $\quad \left([\hat{Q},\hat{H}]=0\right)$

$$[\hat{Q},\hat{X}]=q_X\hat{X}$$

charge of $\hat{X}$ $\quad \left(\because \hat{Q}\hat{X}|\Omega\rangle = q_X\hat{X}|\Omega\rangle\right) \quad \hat{Q}|\Omega\rangle = 0$

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$\langle b| \dots |a\rangle \;\Downarrow$

$$(q_b - q_a - q_X)\langle b|\hat{X}|a\rangle = 0$$

$$\hat{Q}|a\rangle = q_a|a\rangle$$

$\Rightarrow$ selection rule: $\langle b|\hat{X}|a\rangle \neq 0 \Longrightarrow q_b - q_a - q_X = 0$

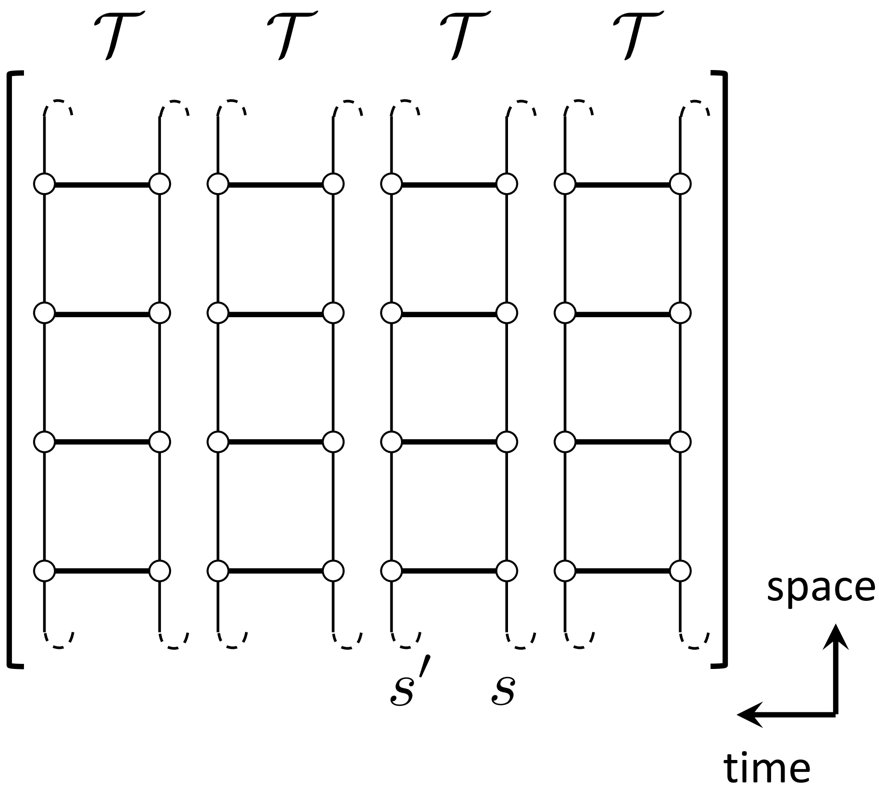
$\Rightarrow$ for $b = 0 = \Omega \quad \langle\Omega|\hat{X}|a\rangle \neq 0 \Longrightarrow q_a = q_X$

For discrete symmetry $\hat{D}\hat{X}\hat{D}^{-1} = q_X\hat{X} \qquad \langle\Omega|\hat{X}|a\rangle \neq 0 \Longrightarrow q_a q_X = 1$

$\pm$

TM + Tensor network

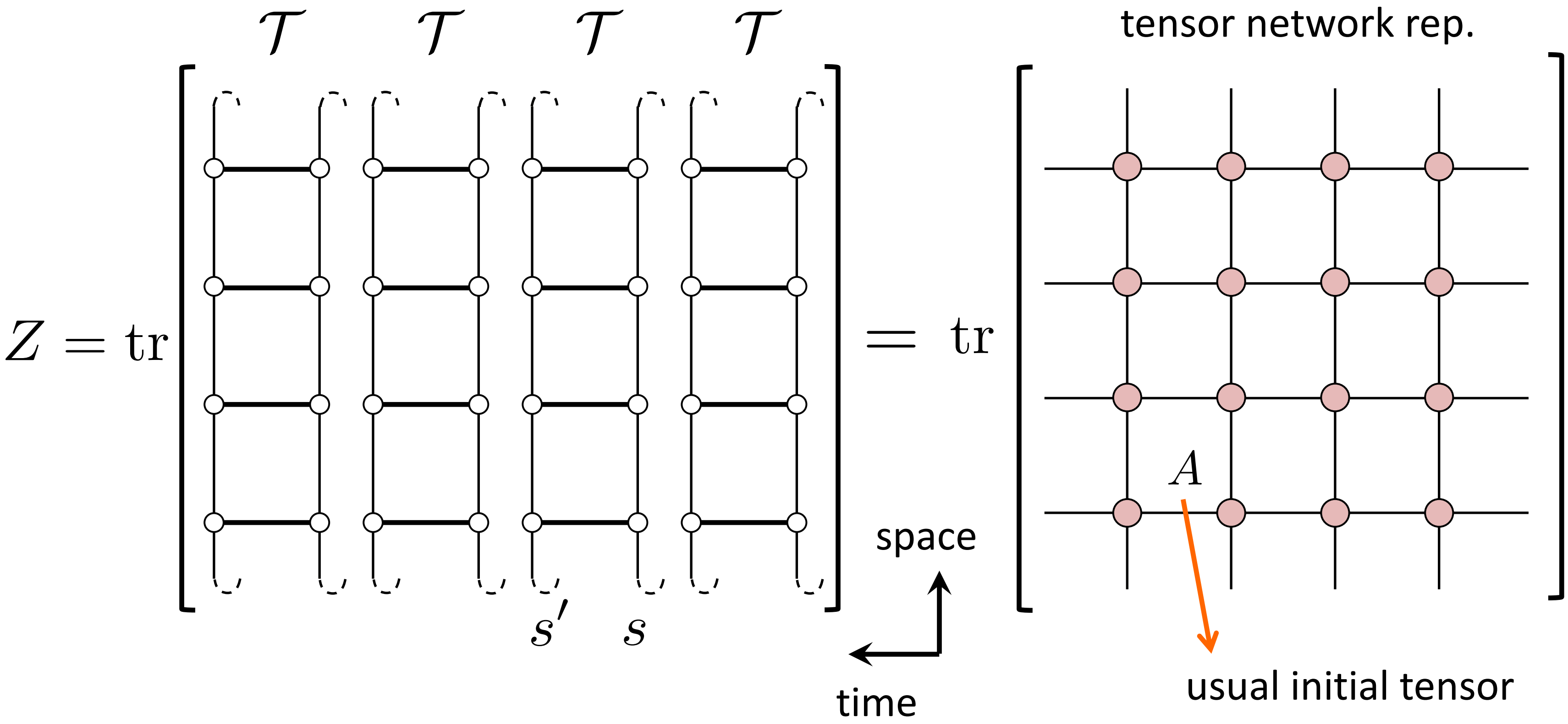
$$Z = \mathrm{tr}$$



$$\mathcal{T}_{s's} = \left(\prod_x e^{\beta s'_x s_x}\right)\left(\prod_x e^{\frac{1}{2}\beta s'_{x+1} s'_x + \frac{1}{2}\beta s_{x+1} s_x}\right)$$

for 2D Ising

TM + Tensor network



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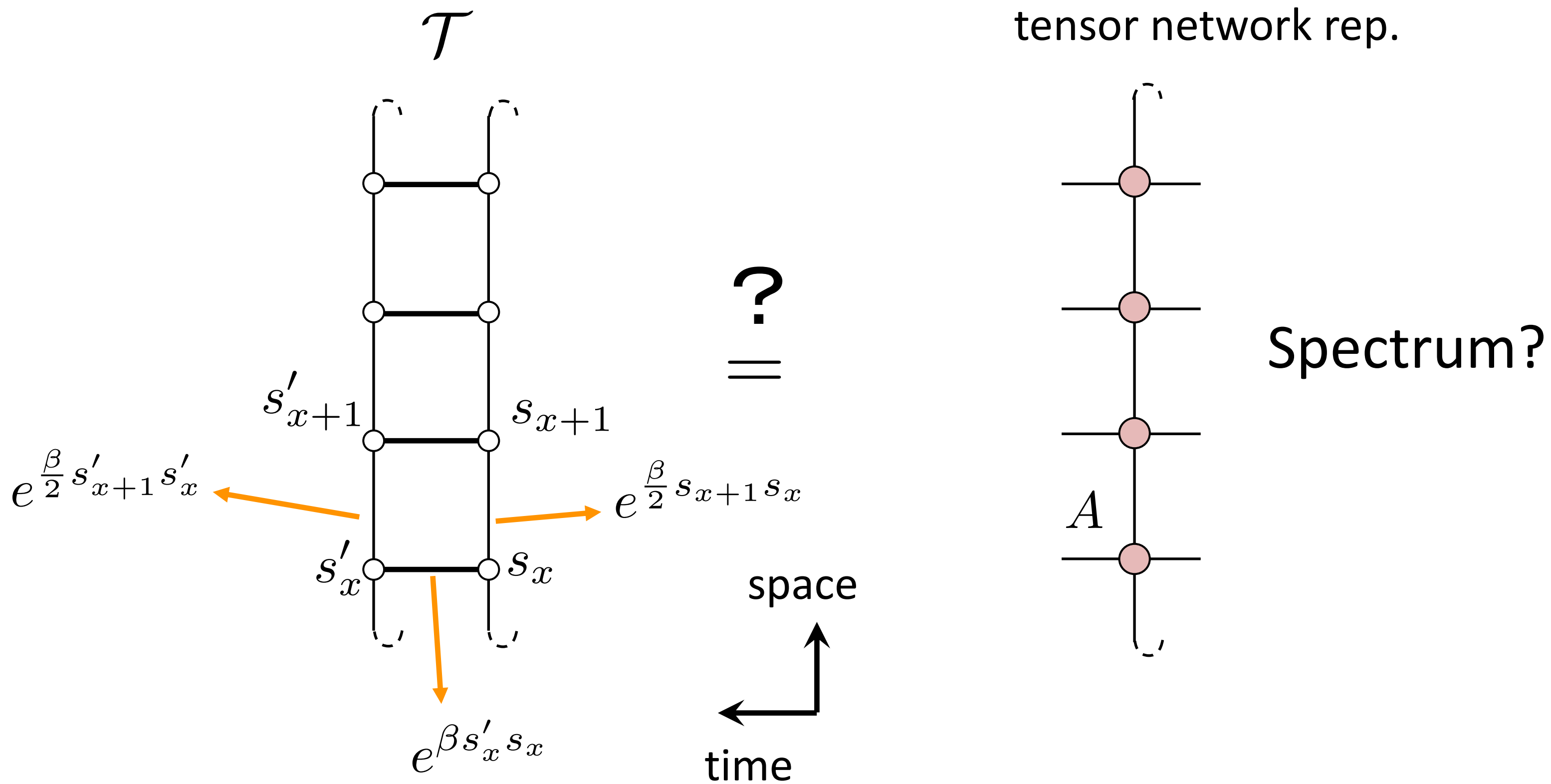

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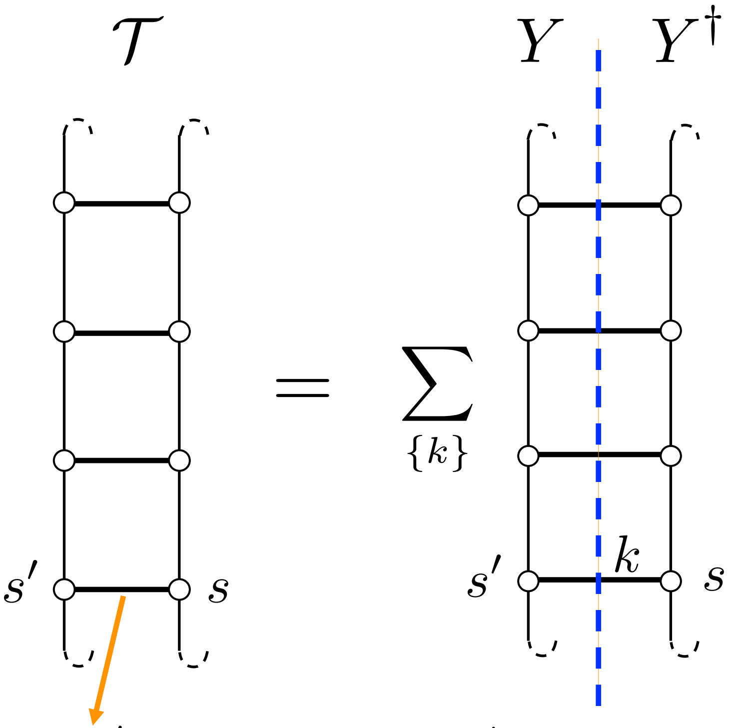
for 2D Ising

TM + Tensor network

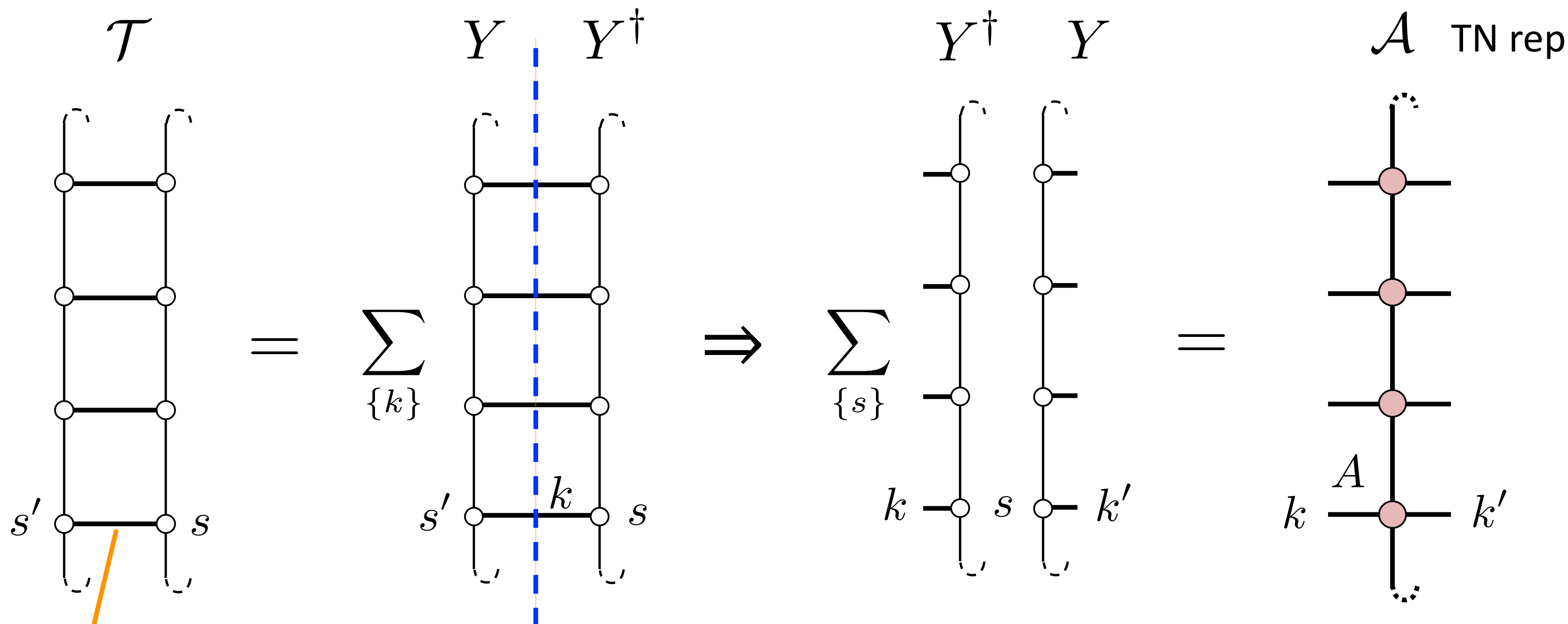
$$Z = \mathrm{tr}[\mathcal{T}^n] \qquad = \mathrm{tr}[(YY^\dagger)^n]$$



$$e^{\beta s' s} \underset{\text{EVD}}{=} \sum_k u_{s'k} \lambda_k u^\dagger_{ks}$$

TM + Tensor network

$$Z = \mathrm{tr}[\mathcal{T}^n] \quad = \mathrm{tr}[(YY^\dagger)^n] \quad = \mathrm{tr}[(Y^\dagger Y)^n] \quad = \mathrm{tr}[\mathcal{A}^n]$$



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EVD

TM + Tensor network

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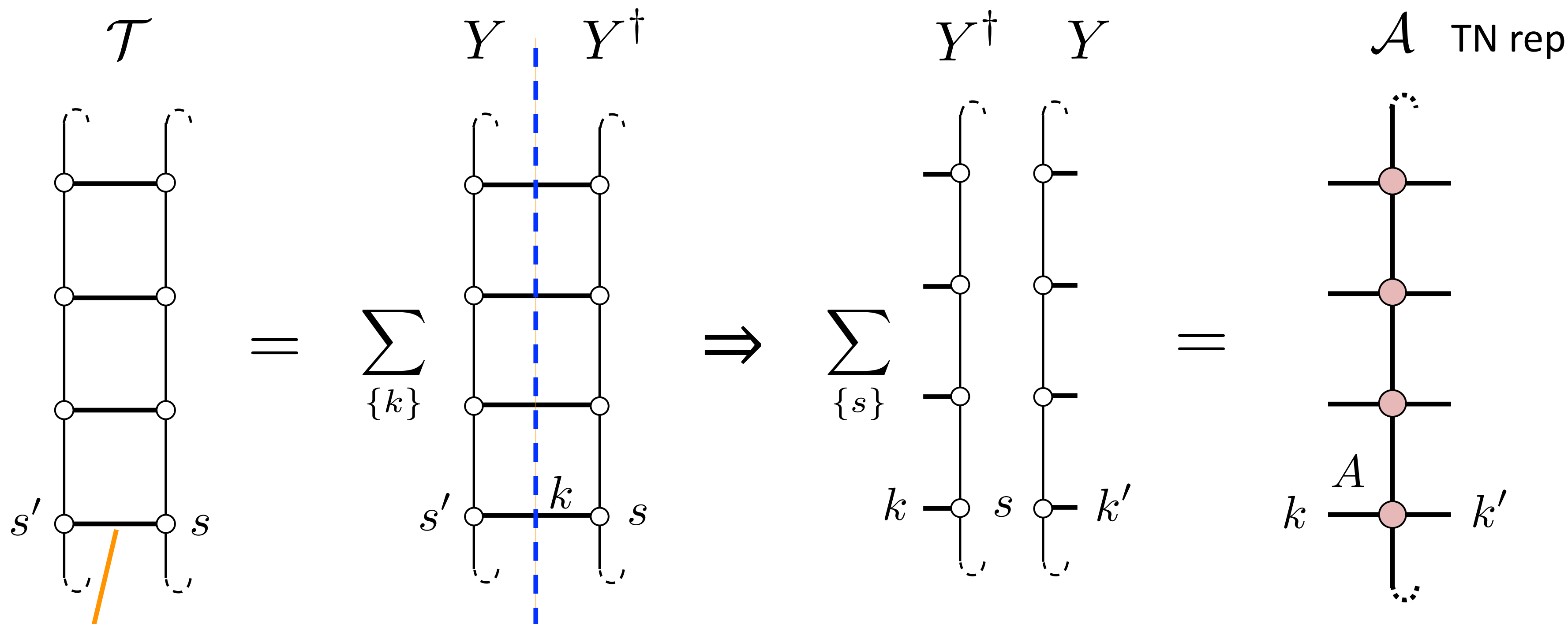


$$e^{\beta s' s} \overset{\text{EVD}}{=} \sum_k u_{s'k} \lambda_k u^\dagger_{ks}$$

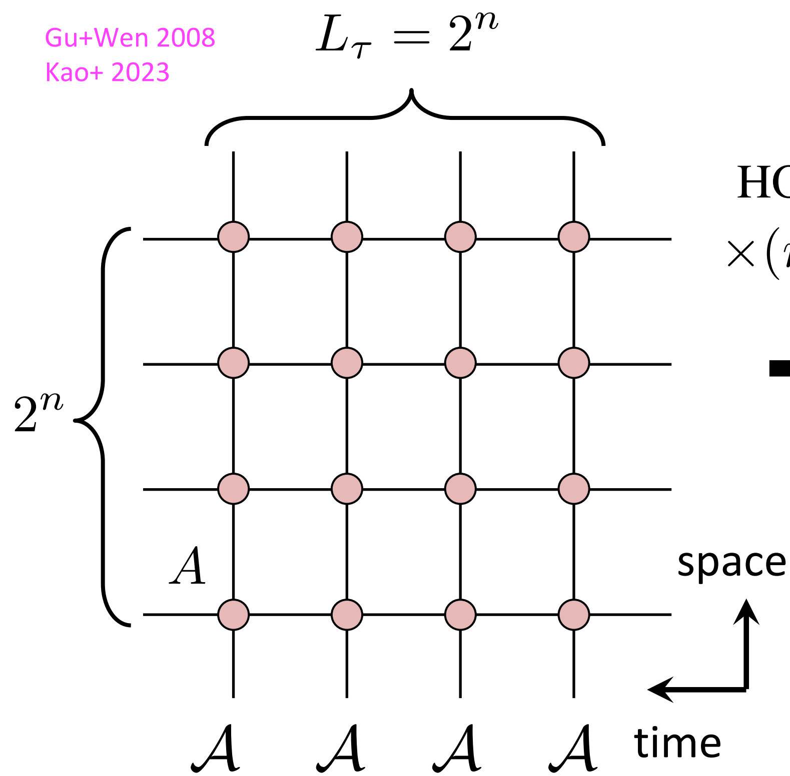
$$Y \overset{\text{SVD}}{=} U \sigma W^\dagger$$

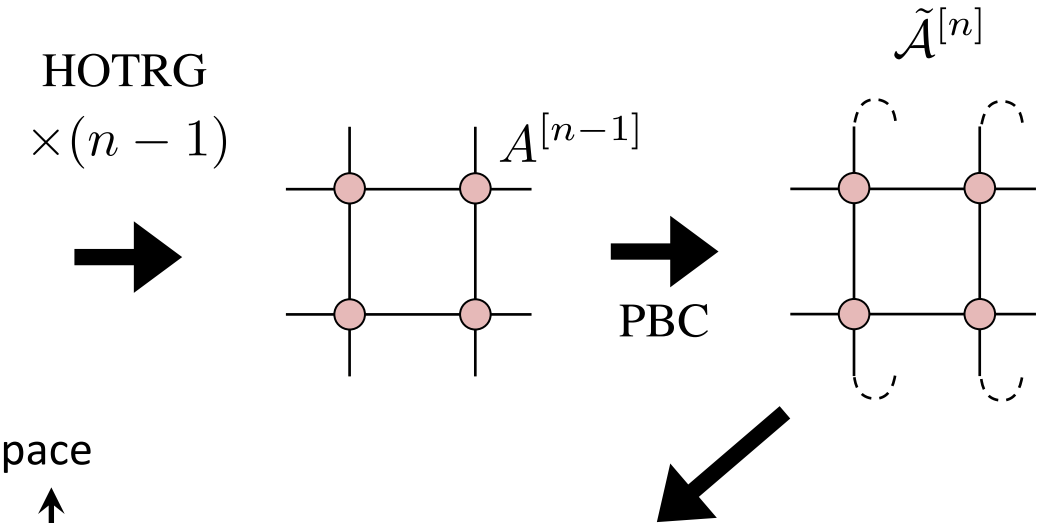
$$\mathcal{T} = YY^\dagger = U\underline{\sigma^2}U^\dagger \qquad \sigma^2 \to e^{-\omega}$$

$$Y^\dagger Y = \mathcal{A} = W\underline{\sigma^2}W^\dagger$$

Identical spectrum!

How to obtain spectrum using HOTRG





diagonalization

$$\tilde{\mathcal{A}}^{[n]} = W^{[n]} \lambda^{[n]} W^{[n]\dagger}$$

$$(\mathcal{T} = U e^{-\omega} U^\dagger)$$

$$\omega_a^{[n]} \equiv \frac{1}{L_\tau} \log \left(\frac{\lambda_0^{[n]}}{\lambda_a^{[n]}} \right) \approx \omega_a$$

$$\tilde{\mathcal{A}}^{[n]} \approx \mathcal{A}^{L_\tau}$$

How to compute matrix elements

$\langle b|\hat{\mathcal{O}}_q|a\rangle = (U^\dagger \mathcal{O}_q U)_{ba}$ $\left(\mathcal{T} = U e^{-\omega} U^\dagger\right)$

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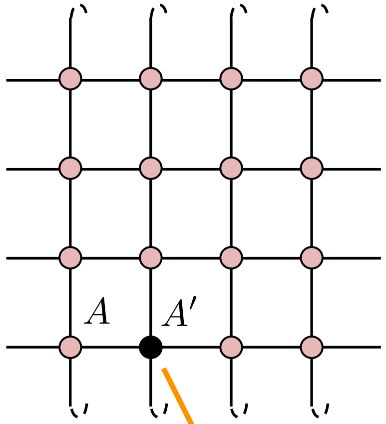
$$\langle b|\hat{\mathcal{O}}_q|a\rangle = (U^\dagger \mathcal{O}_q U)_{ba} \qquad (\mathcal{T} = U e^{-\omega} U^\dagger)$$

$$m = L_\tau/2$$

$$= (U^\dagger \underline{\mathcal{T}^{-m}} \underline{\mathcal{T}^m \mathcal{O}_q \mathcal{T}^{m+1}} \underline{\mathcal{T}^{-(m+1)}} U)_{ba} \qquad (\because \mathcal{T}\mathcal{T}^{-1} = 1)$$

$$\mathcal{T}^{-1} = U e^{+\omega} U^\dagger$$

roughly speaking



one-point function

impurity tensor

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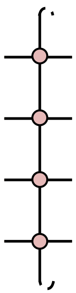
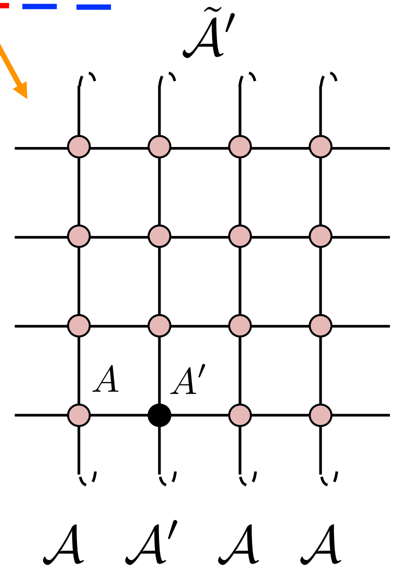
$$\mathcal{T} = YY^\dagger \qquad \mathcal{T}^{-1} = U e^{+\omega} U^\dagger$$

$$= e^{\omega(m-1/2)} W^\dagger \tilde{\mathcal{A}}' W e^{\omega(m+1/2)}$$



$$= \quad \mathcal{A} = W e^{-\omega} W^\dagger$$



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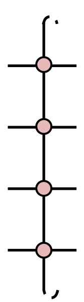
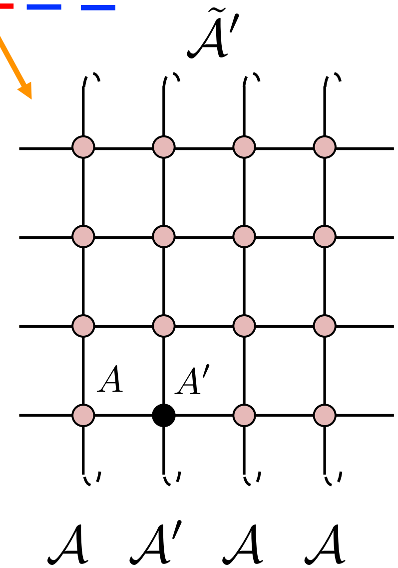


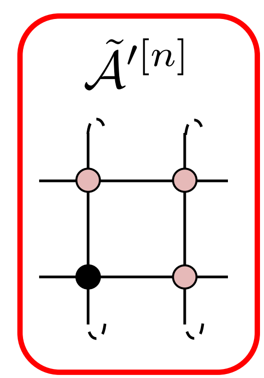
$$= \quad \mathcal{A} = W e^{-\omega} W^\dagger$$

HOTRG

$$\omega^{[n]}, \quad W^{[n]}$$



HOTRG



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$$= e^{\omega(m-1/2)} W^\dagger \tilde{\mathcal{A}}' W e^{\omega(m+1/2)}$$

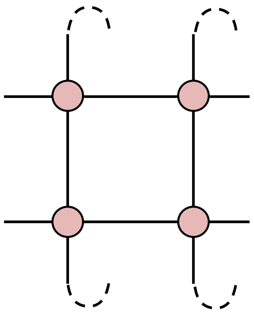
$$\approx e^{\omega^{[n]}(m-1/2)} W^{[n]\dagger} \tilde{\mathcal{A}}'^{[n]} W^{[n]} e^{\omega^{[n]}(m+1/2)}$$

all building blocks are computed from tensor network representations

Numerical results for 2D Ising model

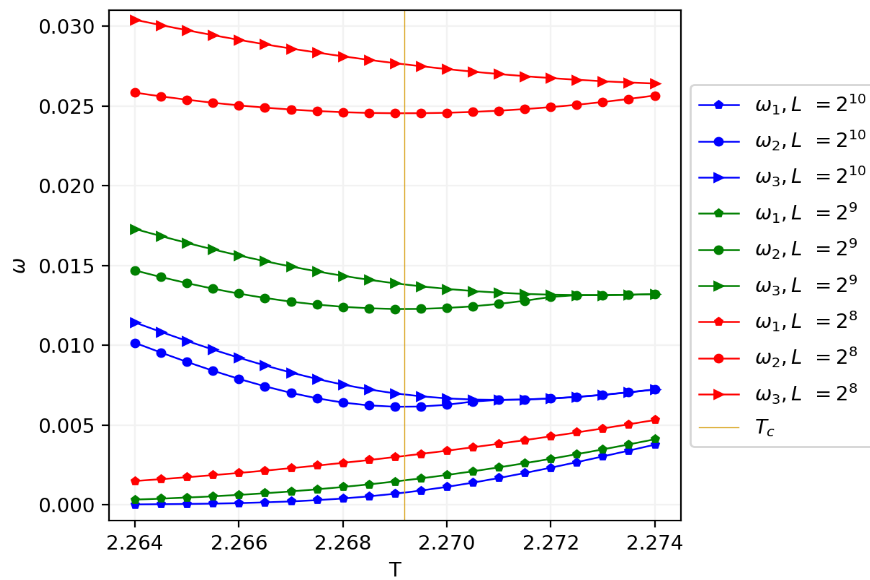
Energy spectrum

after *n* HOTRG steps



$\tilde{\mathcal{A}}^{[n]}$

$$= W^{[n]} \lambda^{[n]} W^{[n]\dagger} \quad \rightarrow \quad \omega_a^{[n]} \equiv \frac{1}{L_\tau} \log \left(\frac{\lambda_0^{[n]}}{\lambda_a^{[n]}} \right) \approx \omega_a$$



$\chi = 100$

$L = L_\tau$

Energy spectrum

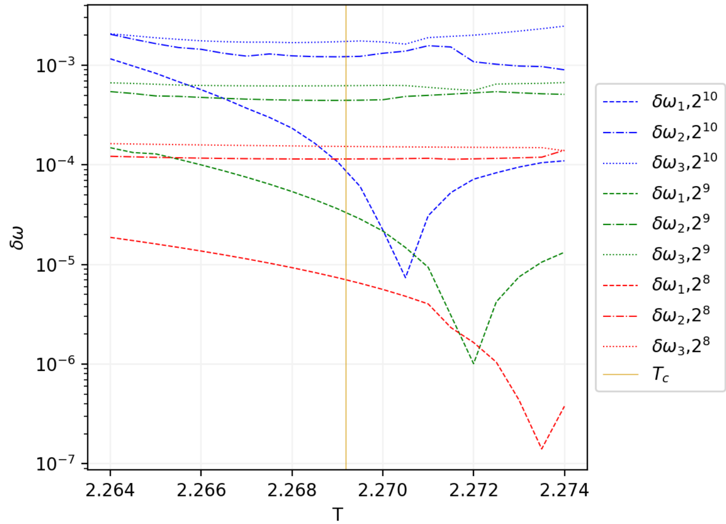
Kaufmann 1949

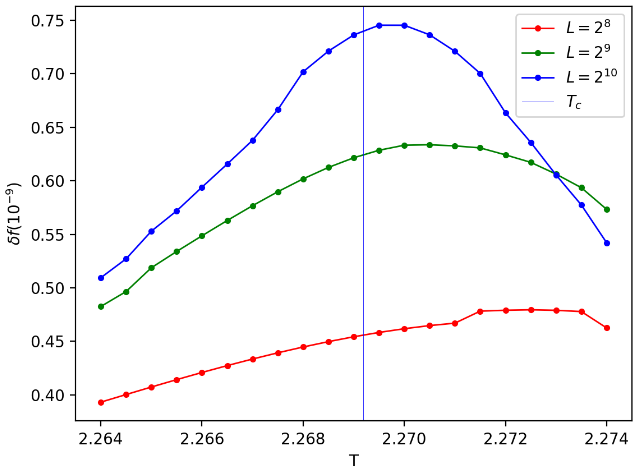
$$\delta\omega = \left|\frac{\omega - \omega_{\mathrm{exact,L}}}{\omega_{\mathrm{exact,L}}}\right|$$

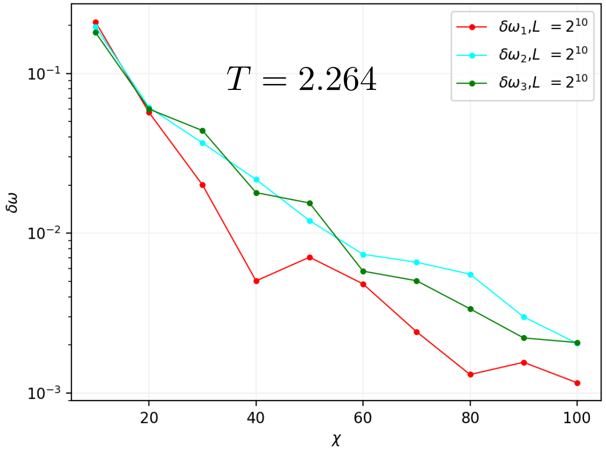
$\chi = 100$

Kaufmann 1949

$$\delta f = \left|\frac{f - f_{\mathrm{exact,L}}}{f_{\mathrm{exact,L}}}\right|$$





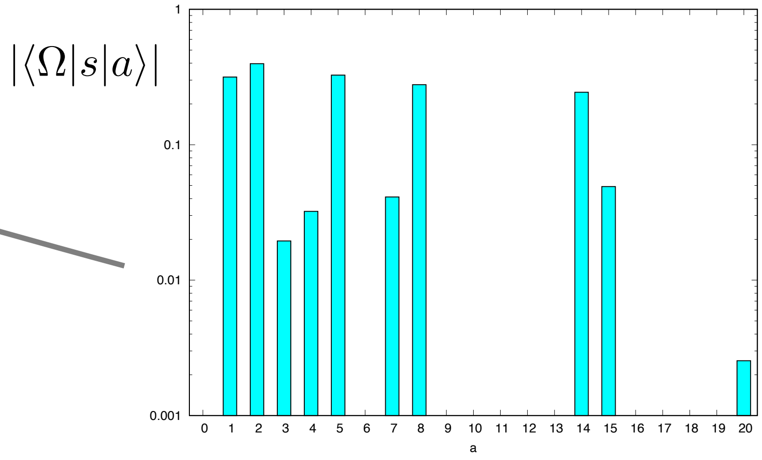


Determination of quantum number

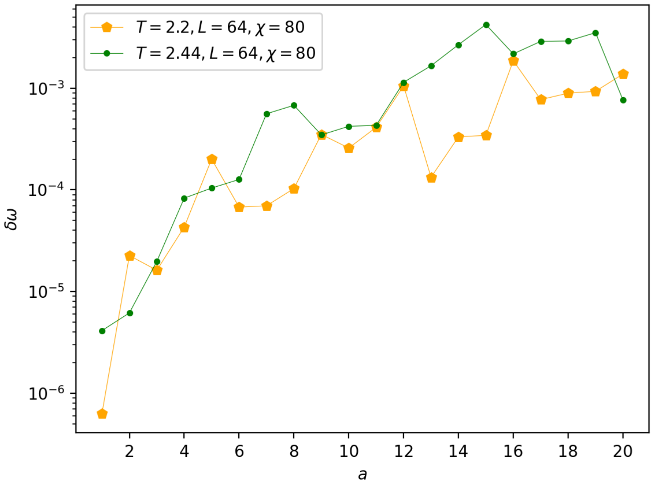
$Z_2 : + \quad - \quad q_a$

$\langle \Omega | s_{x=0} | a \rangle \neq 0 \Longrightarrow q_a = -$

$T = 2.44 \qquad L = 64 \qquad \chi = 80$



a	$\omega^{[\text{exact}]}$	q_a	$\omega^{[\text{hotrg}]}$	q_a	$\delta\omega$
1	0.1262302	−	0.1262307	−	0.000004
2	0.1597880	−	0.1597889	−	0.000006
3	0.1597880	−	0.1597911	−	0.000020
4	0.2326853	−	0.2327046	−	0.000083
5	0.2326853	−	0.2327095	−	0.000104
6	0.2708016	+	0.2708359	+	0.000127
7	0.3181546	−	0.3183329	−	0.000560
8	0.3181546	−	0.3183705	−	0.000679
9	0.3290037	+	0.3291180	+	0.000347
10	0.3290037	+	0.3291425	+	0.000422
11	0.3290037	+	0.3291456	+	0.000431
12	0.3290037	+	0.3293794	+	0.001142
13	0.3872058	+	0.3878486	+	0.001660
14	0.4073042	−	0.4083937	−	0.002675
15	0.4073042	−	0.4090231	−	0.004220
16	0.4100181	+	0.4109090	+	0.002173
17	0.4100181	+	0.4112006	+	0.002884
18	0.4100181	+	0.4112120	+	0.002912
19	0.4100181	+	0.4114574	+	0.003510
20	0.4457831	−	0.4461242	−	0.000765



Bethe-Salpeter wave function

$T = 2.44 \quad L = 64 \quad \chi = 80$

$$\psi_a(x) = \langle \Omega | s_x | a \rangle$$

$$(0 \le x < L)$$

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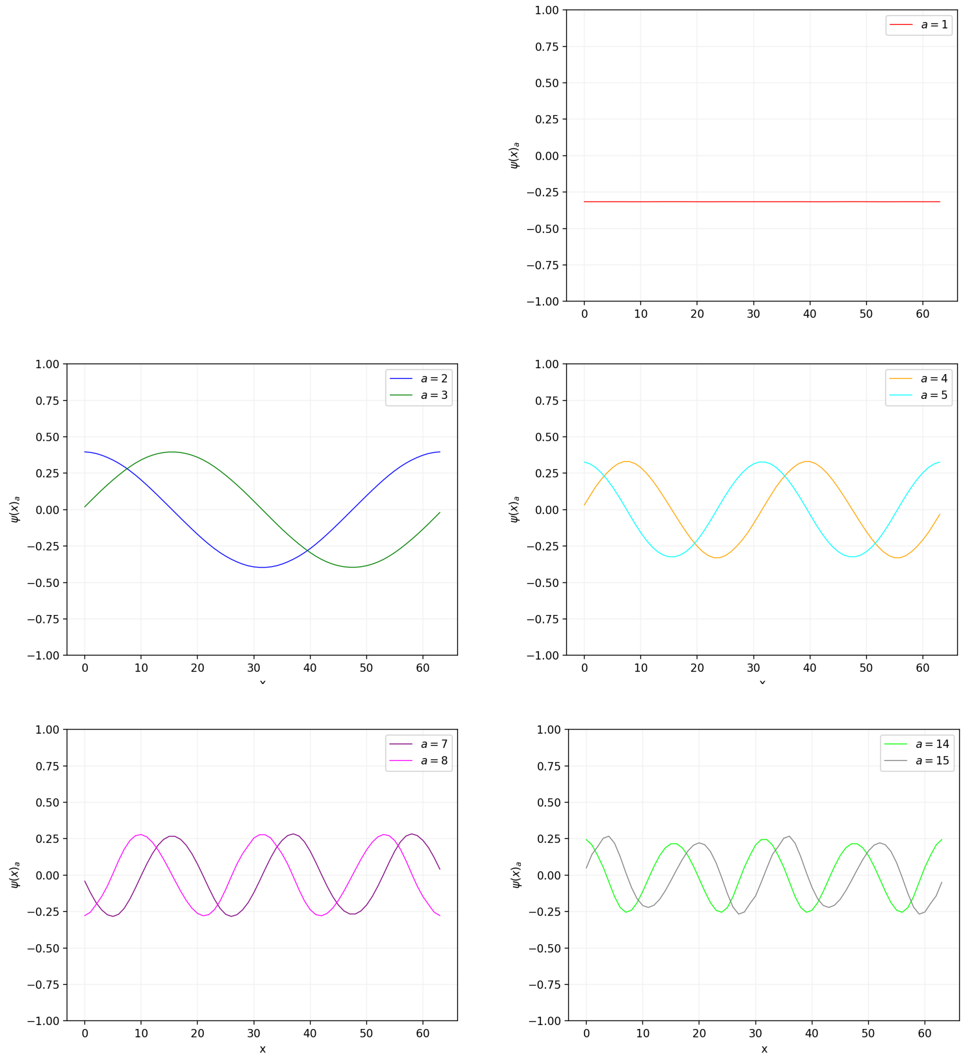

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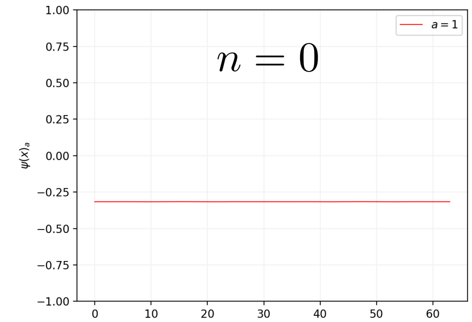
$$(0 \le x < L) \qquad p = \frac{2\pi}{L} n \quad (n = 0, 1, 2, \cdots)$$



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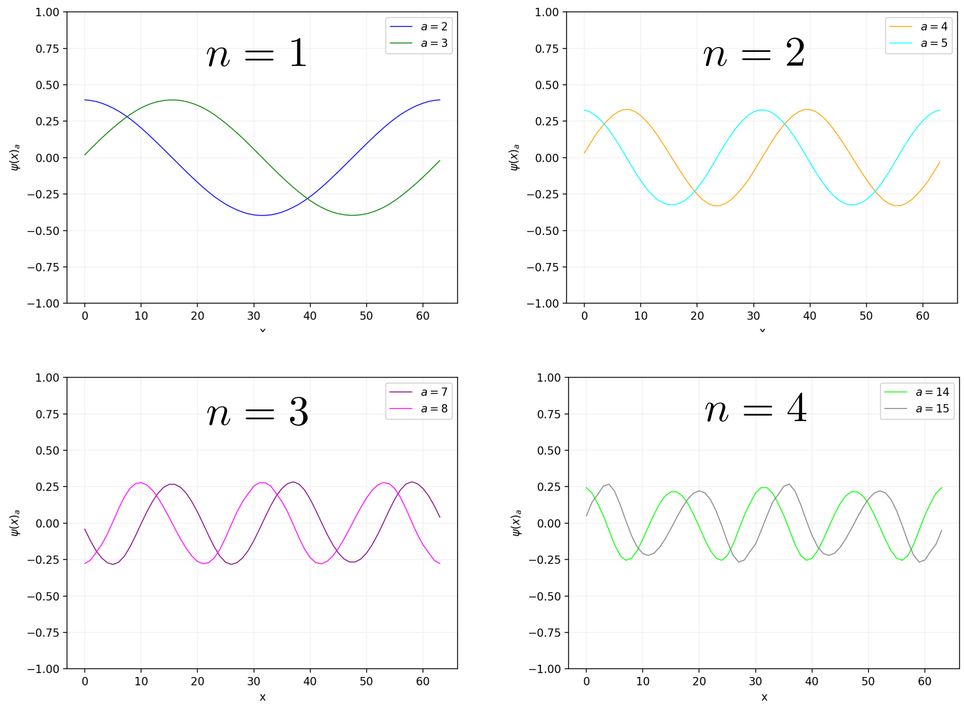


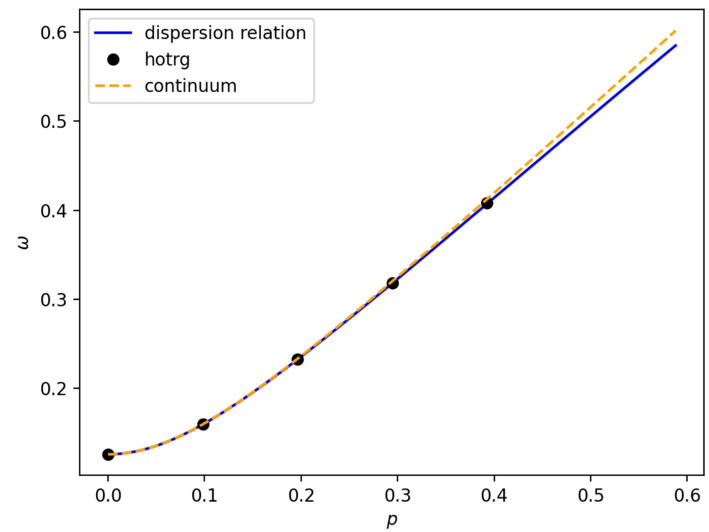
Momentum identification

$T = 2.44 \quad L = 64 \quad \chi = 80$

$$\sum_x e^{-ipx} \langle \Omega | s_x | a \rangle \neq 0 \Longrightarrow p : \text{momentum of } |a\rangle$$

a	$\omega^{[\text{exact}]}$	q_a	$\omega^{[\text{hotrg}]}$	q_a	$\lvert p \rvert$
1	0.1262302	$-$	0.1262307	$-$	0
2	0.1597880	$-$	0.1597889	$-$	$2\pi/L$
3	0.1597880	$-$	0.1597911	$-$	$2\pi/L$
4	0.2326853	$-$	0.2327046	$-$	$4\pi/L$
5	0.2326853	$-$	0.2327095	$-$	$4\pi/L$
6	0.2708016	$+$	0.2708359	$+$	$*$
7	0.3181546	$-$	0.3183329	$-$	$6\pi/L$
8	0.3181546	$-$	0.3183705	$-$	$6\pi/L$
9	0.3290037	$+$	0.3291180	$+$	$*$
10	0.3290037	$+$	0.3291425	$+$	$*$
11	0.3290037	$+$	0.3291456	$+$	$*$
12	0.3290037	$+$	0.3293794	$+$	$*$
13	0.3872058	$+$	0.3878486	$+$	$*$
14	0.4073042	$-$	0.4083937	$-$	$8\pi/L$
15	0.4073042	$-$	0.4090231	$-$	$8\pi/L$
16	0.4100181	$+$	0.4109090	$+$	$*$
17	0.4100181	$+$	0.4112006	$+$	$*$
18	0.4100181	$+$	0.4112120	$+$	$*$
19	0.4100181	$+$	0.4114574	$+$	$*$
20	0.4457831	$-$	0.4461242	$-$	0

"$q = -$" sector only



lattice: $\omega(p) = \cosh^{-1}(1 - \cos p + \cosh m)$

continuum: $\omega(p) = \sqrt{m^2 + p^2}$

Summary

- We develop a spectroscopy scheme of Lagrangian tensor network approach
- Quantum number of eigen state is judged by looking at associated matrix elements (one-point function)
- BS wave function for low-lying states can be computed
- Relatively higher momentum states and dispersion relation are clearly observed

Future

- Application to other quantum field theories
- Two-particle state
- Scattering phase shift

Backup slides

How to compute matrix elements

$$\langle b|\hat{\mathcal{O}}_q|a\rangle = (U^\dagger \mathcal{O}_q U)_{ba} \qquad (\mathcal{T} = U e^{-\omega} U^\dagger)$$

$$m = L_\tau/2$$

$$= (U^\dagger \mathcal{T}^{-m} \mathcal{T}^{m} \mathcal{O}_q \mathcal{T}^{m+1} \mathcal{T}^{-(m+1)} U)_{ba} \qquad (\because \mathcal{T}\mathcal{T}^{-1} = 1)$$

$$\mathcal{T} = YY^\dagger \qquad \mathcal{T}^{-1} = U e^{+\omega} U^\dagger$$

$$= (U^\dagger (U e^{\omega} U^\dagger)^m (YY^\dagger)^m \mathcal{O}_q (YY^\dagger)^{m+1} (U e^{\omega} U^\dagger)^{m+1} U)_{ba}$$

$$= (e^{m\omega} U^\dagger Y (Y^\dagger Y)^{m-1} Y^\dagger \mathcal{O}_q Y (Y^\dagger Y)^m Y^\dagger U e^{(m+1)\omega})_{ba}$$

$$e^{-\omega/2} W^\dagger \qquad \mathcal{A}' \qquad \mathcal{A} \qquad W e^{-\omega/2} \qquad (\because Y = U e^{-\omega/2} W^\dagger)$$

$$= (e^{(m-1/2)\omega} W^\dagger \mathcal{A}^{m-1} \mathcal{A}' \mathcal{A}^m W e^{(m+1/2)\omega})_{ba}$$

$$\tilde{\mathcal{A}}'$$