Towards tensor renormalization group study of lattice QCD

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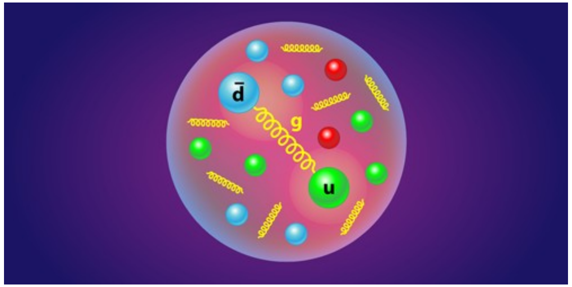
TENSOR NETWORK 2024

Grant-in-Aid for Transformative Research Areas A
Extreme Universe

Lattice QCD

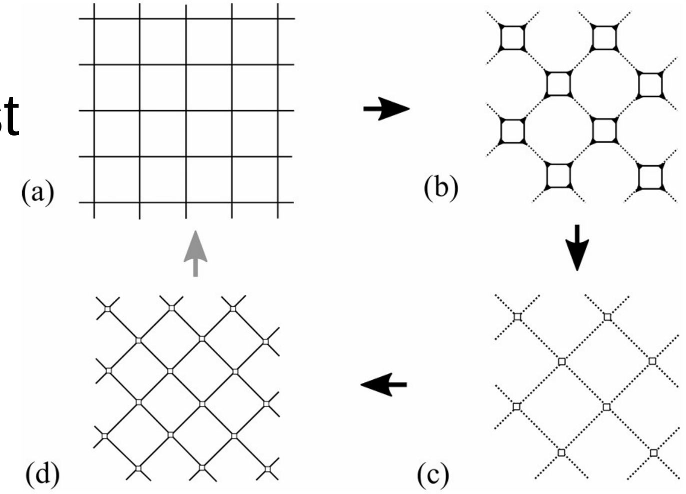
- non-perturbative formulation for quantum chromodynamics
- 4D Euclidean SU(3) gauge theory with Nf=2,3
  - Higher dimensions, Non-abelian, Multiple flavors
- MC computation suffers from the sign problem at finite θ and finite density
- MC computation also suffers from the topology freezing problem toward continuum limit



So far, we do not have a universal method that avoids all the problems...

Tensor renormalization group (TRG)

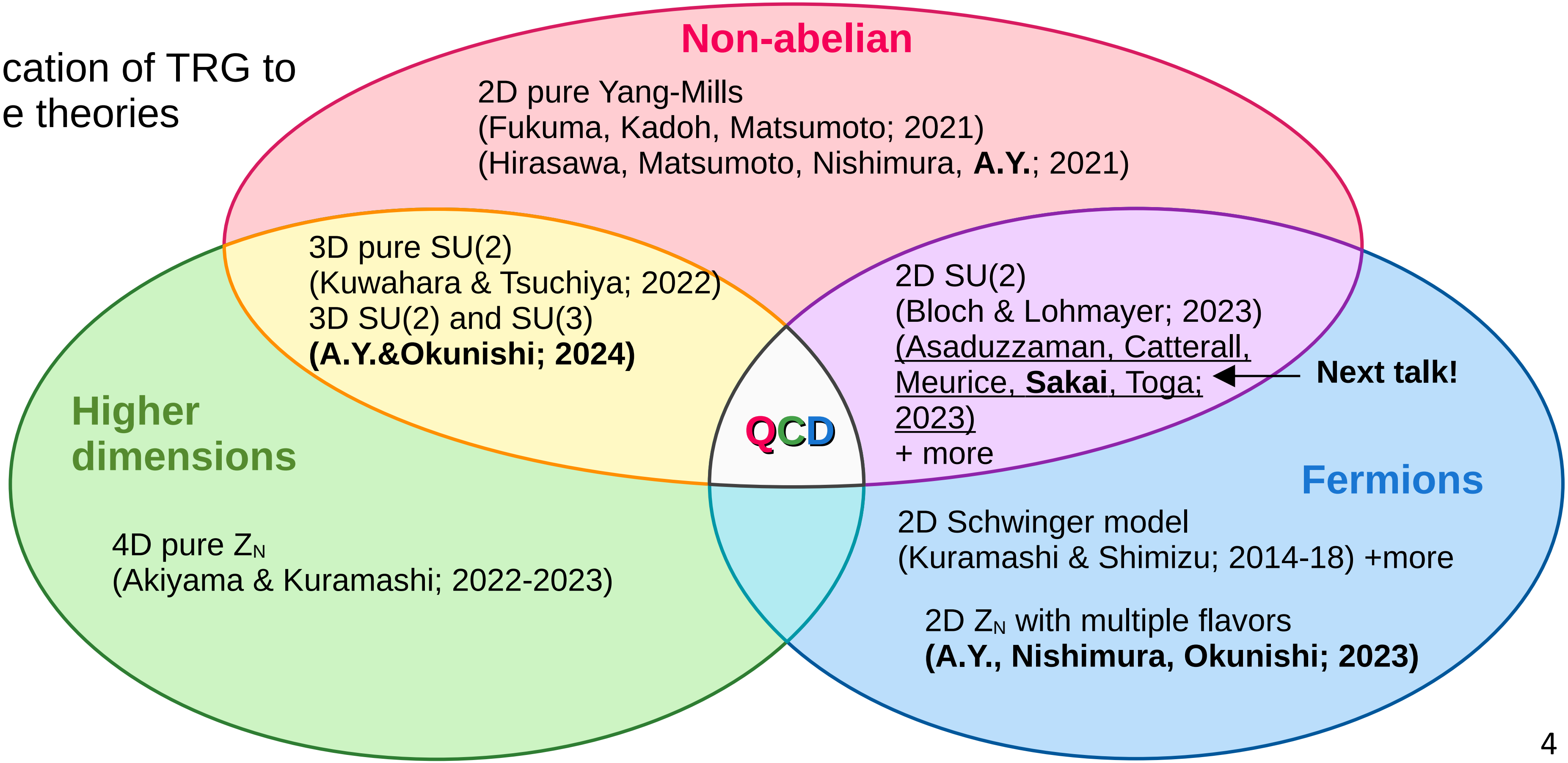
- An alternative to Monte Carlo methods based on coarse graining
- No sampling = No sign problem
- Can access large volumes with log cost
- Can handle fermion/Grassmann numbers directly; Grassmann TRG



[Figures from Okunishi-Nishino-Ueda; 2022]

Progress toward TRG study of Lattice QCD

Application of TRG to gauge theories



Challenges

- TRG can be challenging when the local Hilbert space is large
- By that, I meant QCD
  - Multiple fermion flavors ==> dimension ~ exp(kN_f)
  - Non-abelian gauge symmetry ==> Redundancy in the TN

I will talk about my works on these two directions.

Outline

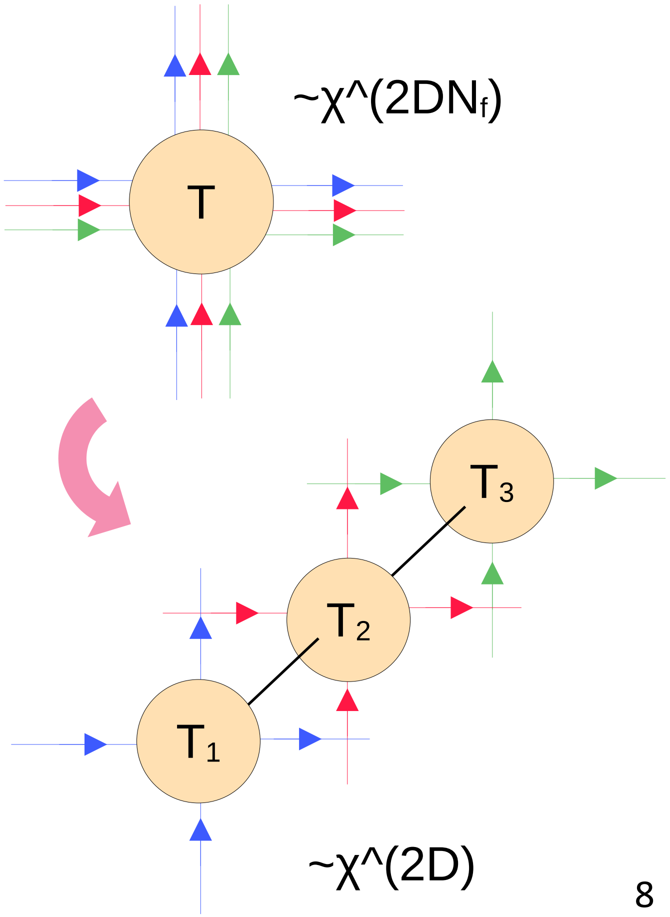
- Part I: Multi-flavor gauge theory
  - The multi-layer formulation
  - Initial tensor compression
  - Result: *finite density* N_f - flavor 2D Z_N theory
- Part II: Armillary sphere formulation
  - Degeneracy in the tensor network
  - Reduced tensor network formulation
  - Result: 3D SU(2) & SU(3) thoery

Part I: Multi-layer construction for multi-flavor gauge theory

Based on [JHEP11(2023)187], with **Jun Nishimura** (KEK) and **Kouichi Okunishi** (Niigata U)

Multi-flavor gauge theory

- The number of tensor legs increases with the number of flavors
- This makes it difficult to consider multi-flavor theory in the tensor network
- This can be avoided by separating flavor d.o.f. from each other



Similar ideas:
- Domain wall fermions (flavor = extra dim)
- MPO-like decomposition (Akiyama; 2023)

Multi-flavor gauge theory

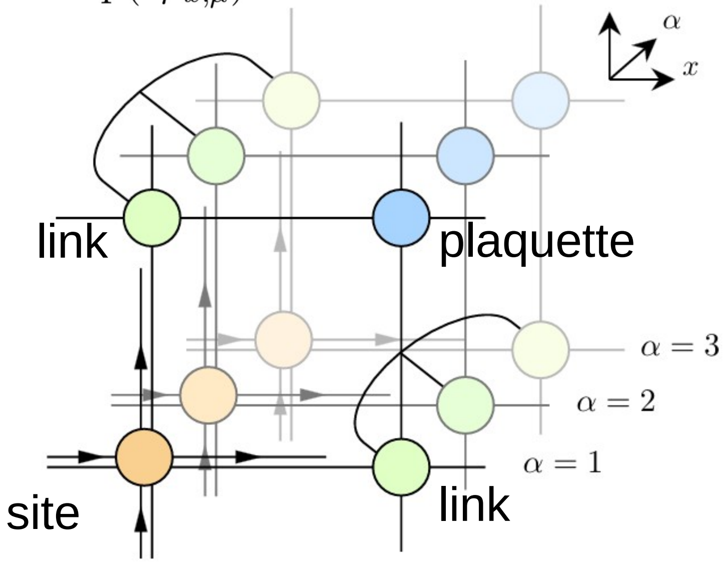
Separate the layers analytically

$$Z = \int D\varphi \prod_{\alpha=1}^{N_\mathrm{f}} \left(D\varphi^{(\alpha)} D\psi^{(\alpha)} D\bar{\psi}^{(\alpha)} \right) \delta(\varphi^{(\alpha)} - \varphi) e^{-\sum_\alpha S^{(\alpha)}}$$

local action for each flavor

$$U_{x,\mu} = \exp(iaA_{x,\mu}) \equiv \exp(i\varphi_{x,\mu})$$

$$S^{(\alpha)} = \frac{1}{N_\mathrm{f}} S_\mathrm{gauge}[\varphi^{(\alpha)}] + \sum_{x \in \Lambda_2} \bar{\psi}_x^{(\alpha)} \not{D}^{(\alpha)} \psi_x^{(\alpha)}$$



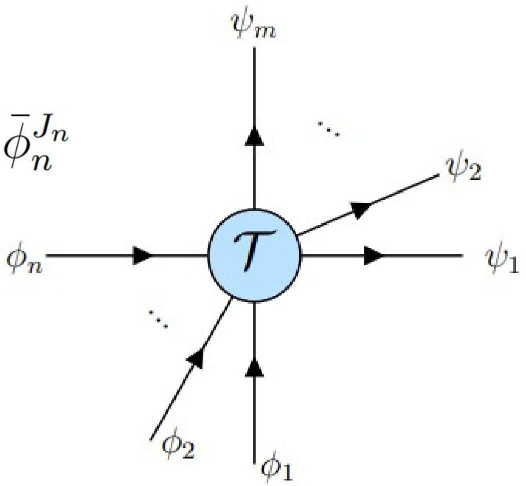
- each layer for each flavor
- connected via delta functions
- Grassmann tensors handle fermions directly

Quick intro: Grassmann tensors

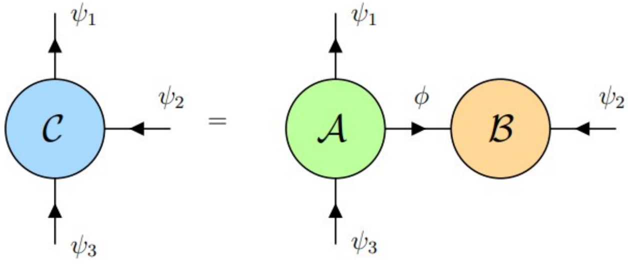
$$\mathcal{T}_{\psi_1\cdots\psi_m\bar{\phi}_1\cdots\bar{\phi}_n} = \sum_{I_1,\cdots,I_m,J_1,\cdots,J_n} T_{I_1\cdots I_m J_1\cdots J_n}\,\psi_1^{I_1}\cdots\psi_m^{I_m}\bar{\phi}_1^{J_1}\cdots\bar{\phi}_n^{J_n}$$

Coefficient tensor

Multi-component Grassmann numbers



Grassmann tensor contraction

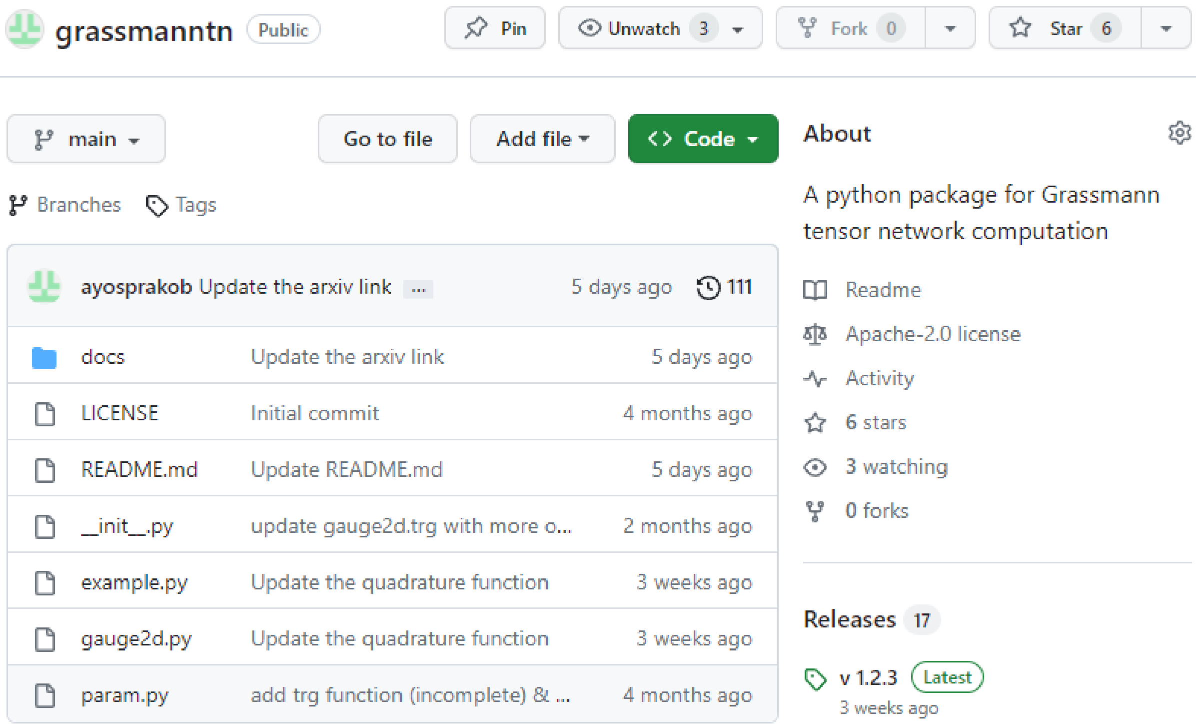


$$C_{IJK} = \sum A_{ILK} B_{JL} s_{JKL}$$

$$s_{JKL} = \sigma_L \times (-)^{p(L)(p(J)+p(K))+p(J)p(K)}$$

GrassmannTN:

a python package for Grassmann TRG / DMRG



Features:
Grassmann contractions,
Tensor reshapes,
SVD/EigVD,
dense/sparse conversions,
Grassmann arithmatic,
Berezin integrals, etc.

complete tutorial for
1+1D Schwinger model (TRG)

https://github.com/ayosprakob/grassmanntn

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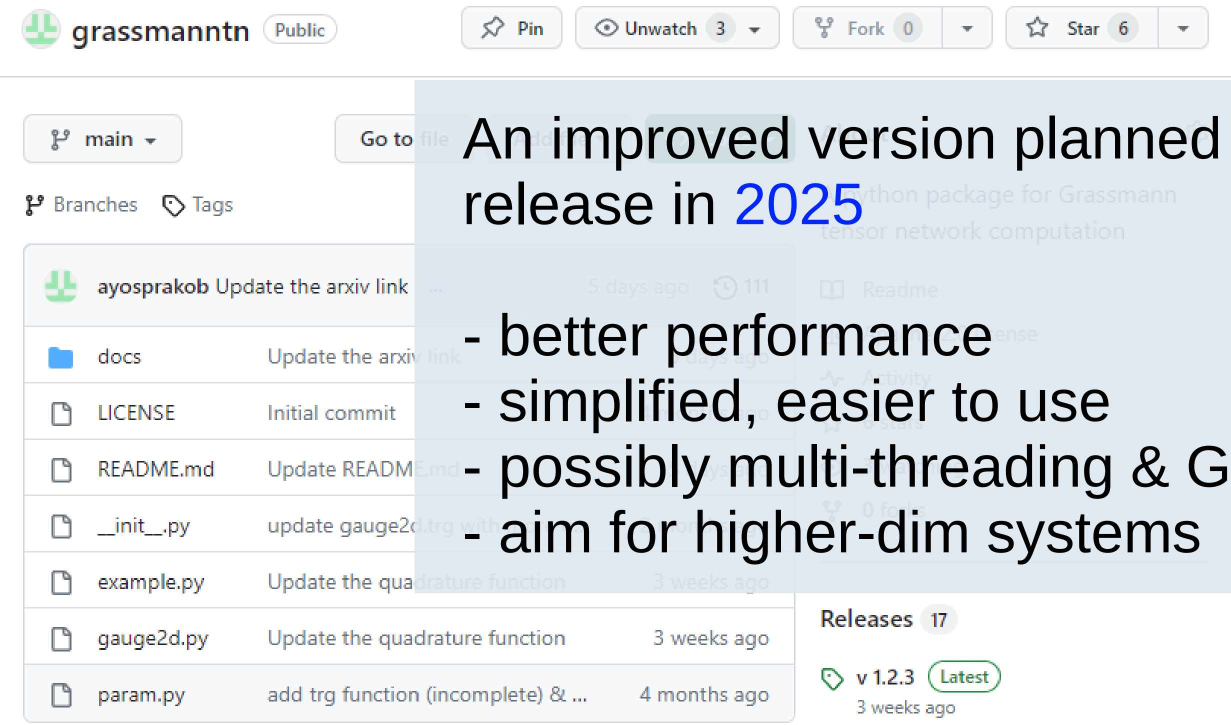
a python package for Grassmann TRG / DMRG



An improved version planned to release in 2025

- better performance
- simplified, easier to use
- possibly multi-threading & GPU
- aim for higher-dim systems

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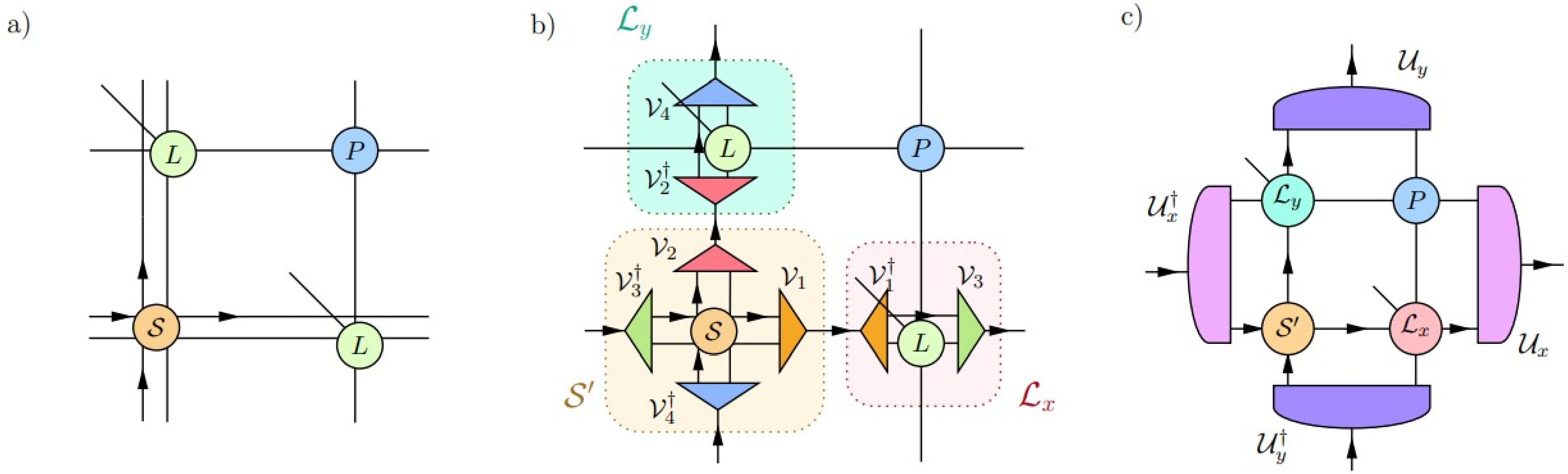
Tensor compression

Proposed compression scheme:



Isometries are first applied around the Grassmann tensor *S*: a → b
Then another set is applied around the whole tensor: b → c

Compression performance

Physical parameters

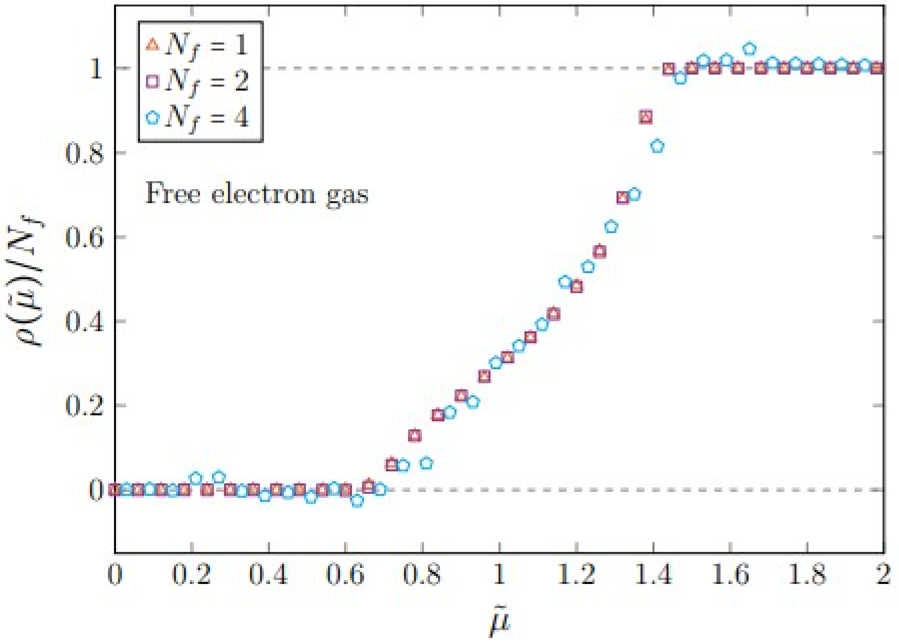
β	$\tilde{\mu}$	N_{f}	K	original size	compressed size	compression ratio	D_x	D_y
0.0	0.0	1	2	67108864	1024	1.53×10^{-5}	4	4
0.0	0.0	1	3	3869835264	2304	5.95×10^{-7}	4	4
0.0	0.0	1	4	68719476736	4096	5.96×10^{-8}	4	4
0.0	0.0	1	5	640000000000	6400	1.00×10^{-9}	4	4
2.0	0.0	1	2	67108864	16384	2.44×10^{-4}	8	8
2.0	0.0	2	2	67108864	16384	2.44×10^{-4}	8	8
2.0	3.0	1	2	67108864	16384	2.44×10^{-4}	8	8
2.0	3.0	2	2	67108864	16384	2.44×10^{-4}	8	8

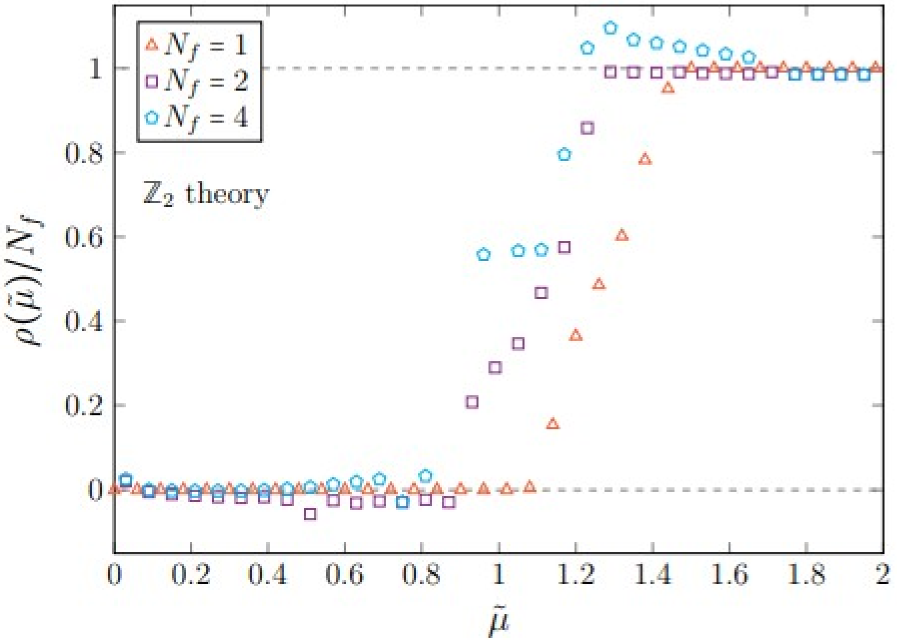
All of these compressions are done without any spectrum truncation!
Such high performance is due to the sparse nature of fermions.

Finite density and Silver Blaze

Flavors: HOTRG (Dcut=64, 32 for Nf=2, 4)
Space-time: Levin-Nave TRG (Dcut=64)

Silver Blaze is reproduced up to $N_f = 4$





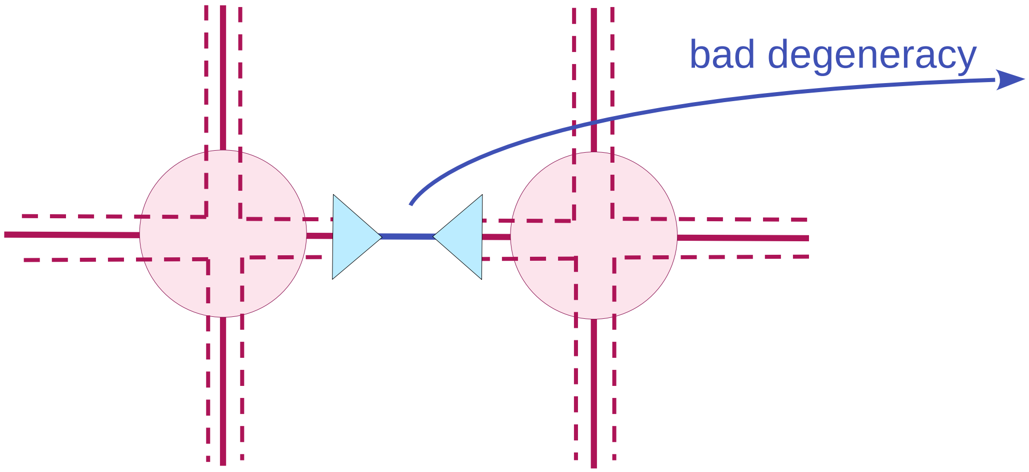
This is typically difficult in Monte Carlo due to the sign problem!

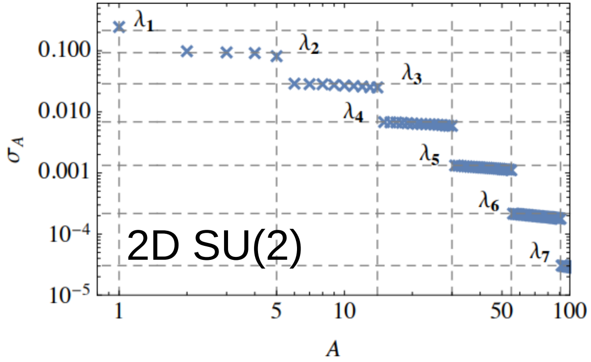
Part II:
Armillary sphere
Non-abelian gauge theory in higher dimensions

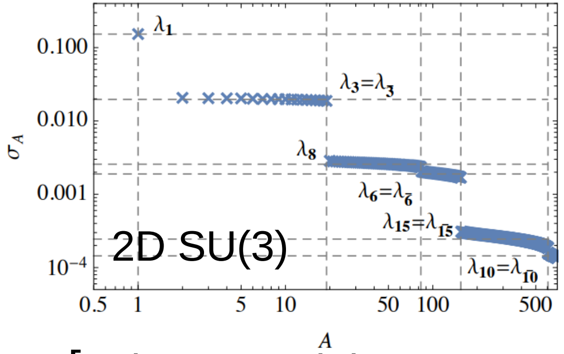
Based on [PTEP 2024 (2024) 7, 073B05] (Formulation)
and [arXiv:2406.16763] (Numerical) with **Kouichi Okunishi** (Niigata U)

Why is non-abelian tensor network difficult?

Internal symmetry in SU(N) is a redundancy in the tensor network that cannot be truncate by an SVD







The entanglement structure is nonlocal...

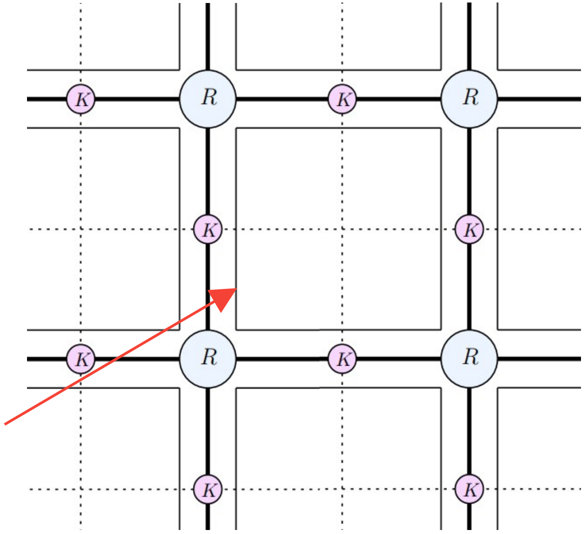
Figures from [Fukuma-Kadoh-Matsumoto; 2021]

Character expansion

- Lesson from 1+1D: the (matrix) index loops can be traced out if we use character expansion

[Hirasawa, Matsumoto, Nishimura, **A.Y.**; 2021]

- Degeneracy is completely eliminated

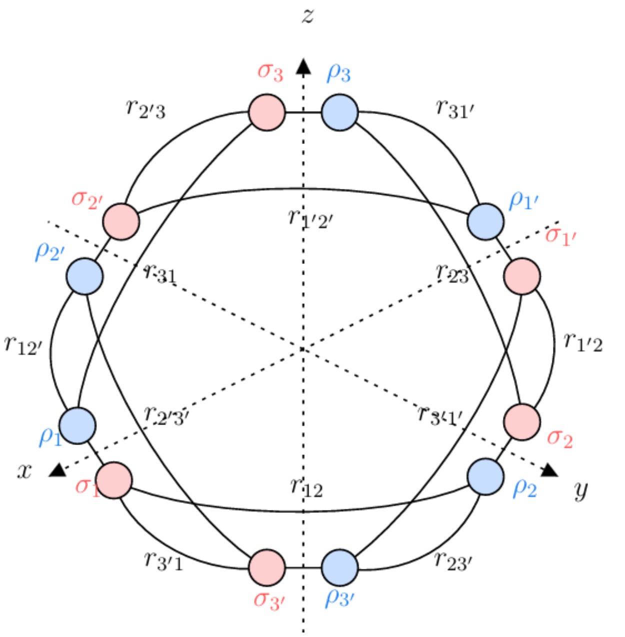


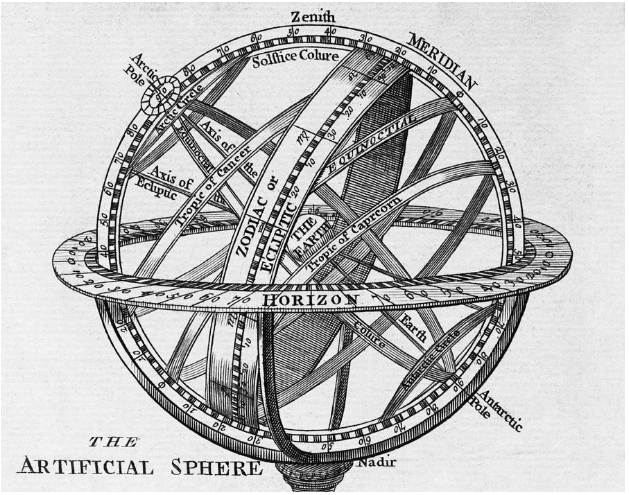
the source of singular value degeneracy

Can we do the same thing for any dimension?

The armillary sphere

Yes! There is a similar closed network in any dimension
Which we call the armillary sphere





armillary sphere
= intersecting circles

This was first noticed by [Oeckl & Pfeiffer;2001] in the context of the spin foam model.

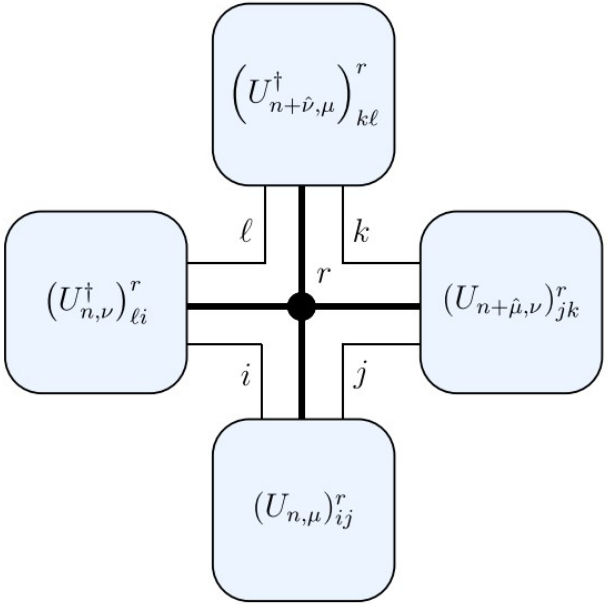
The armillary sphere

Step 1: perform character expansion on the Boltzmann weight

$$e^{\beta \Re \operatorname{tr} P_{n,\mu\nu}} = \sum_r f_r \operatorname{tr}_r(U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu})$$

$$= \sum_r f_r \sum_{i,j,k,l} (U_{n,\mu})^r_{ij} (U_{n+\hat{\mu},\nu})^r_{jk} (U^\dagger_{n+\hat{\nu},\mu})^r_{kl} (U^\dagger_{n,\nu})^r_{li}$$

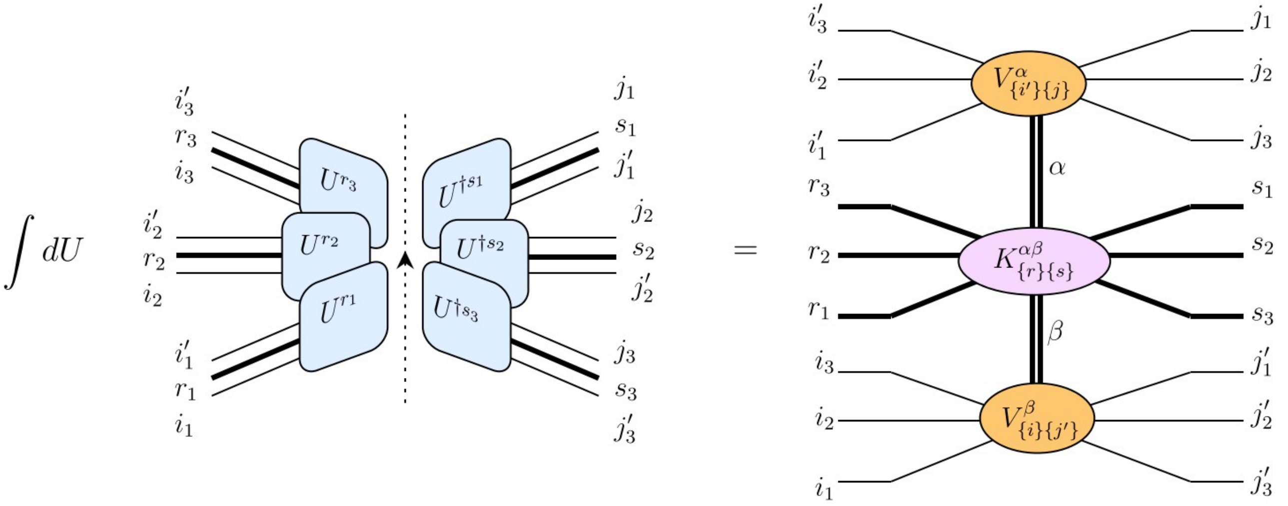
$\operatorname{tr}_r P_{n,\mu\nu} =$



The armillary sphere

Step 2: perform group integral on each link variable

$$\int dU_{n,\mu} (U_{n,\mu})^{r_1}_{i_1 i'_1} \cdots (U_{n,\mu})^{r_{d-1}}_{i_{d-1} i'_{d-1}} (U^\dagger_{n,\mu})^{s_1}_{j_1 j'_1} \cdots (U^\dagger_{n,\mu})^{s_{d-1}}_{j_{d-1} j'_{d-1}} = \sum_{\hat r \in D_{\otimes r}} \sum_{\hat s \in D_{\otimes s}} \sum_{\hat\imath,\hat\jmath} \frac{1}{\dim(\hat r)} C^{\hat r\hat\imath}_{\{r\}\{i\}} C^{\hat r\hat\jmath}_{\{r\}\{i'\}} C^{\hat s\hat\jmath}_{\{s\}\{j\}} C^{\hat s\hat\imath}_{\{s\}\{j'\}} \delta(\hat r \simeq \hat s)$$



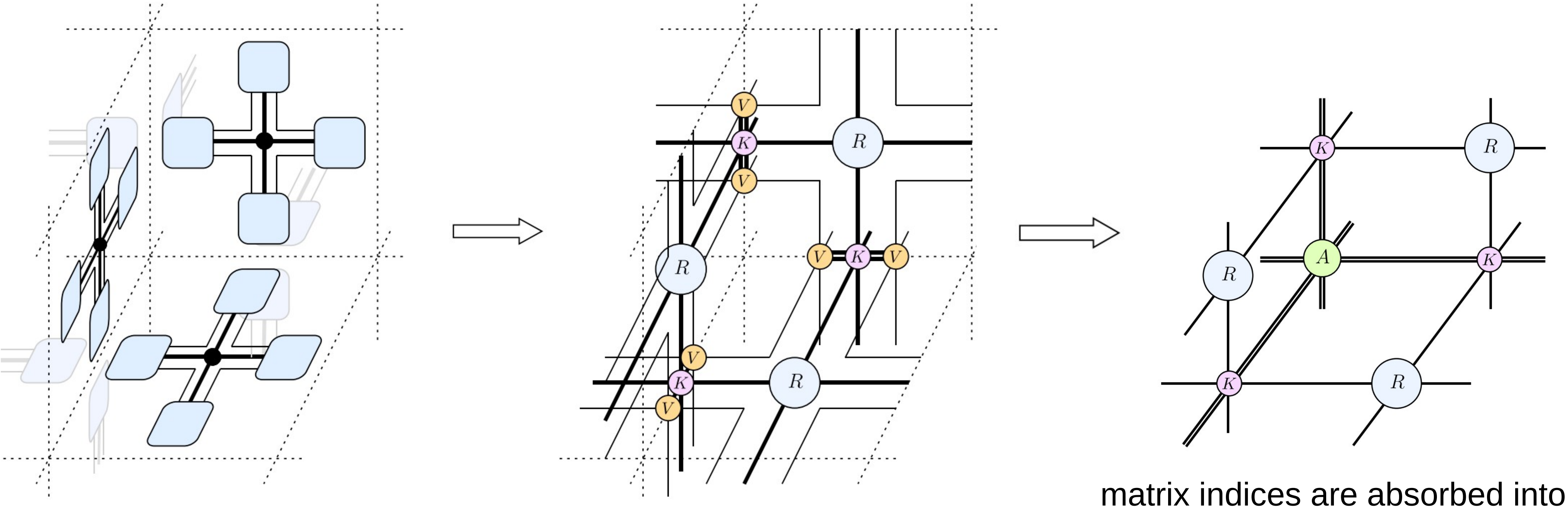
$$V^{\alpha}_{\{i\}\{j'\}} = \delta(\hat r \simeq \hat s) \sum_{\hat\imath} C^{\hat r\hat\imath}_{\{r\}\{i\}} C^{\hat s\hat\imath}_{\{s\}\{j'\}},$$

$$K^{\alpha\beta}_{\{r\}\{s\}} = \frac{1}{\dim(\hat r)} \delta_{\hat r\hat r'} \delta_{\hat s\hat s'} \delta(\hat r \in D_{\otimes r}) \delta(\hat s \in D_{\otimes s});$$

Note: matrix indices (thin lines) are neatly separated into two layers

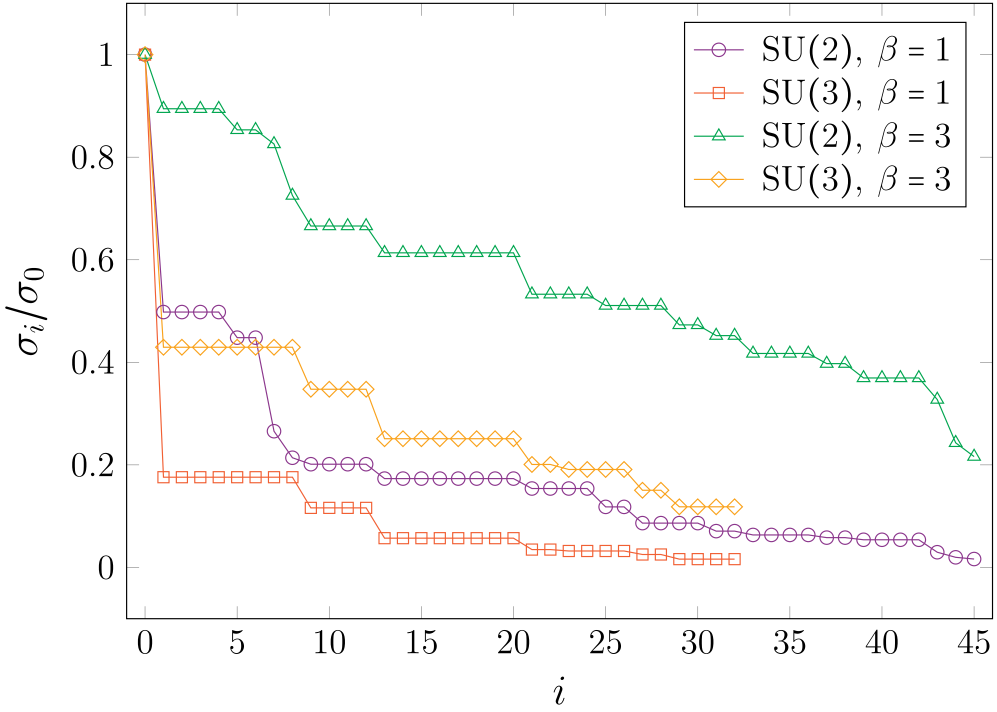
The armillary sphere

Step 3: Contract the matrix indices



matrix indices are absorbed into the armillary tensor ‘A’

Result: singular value spectrum



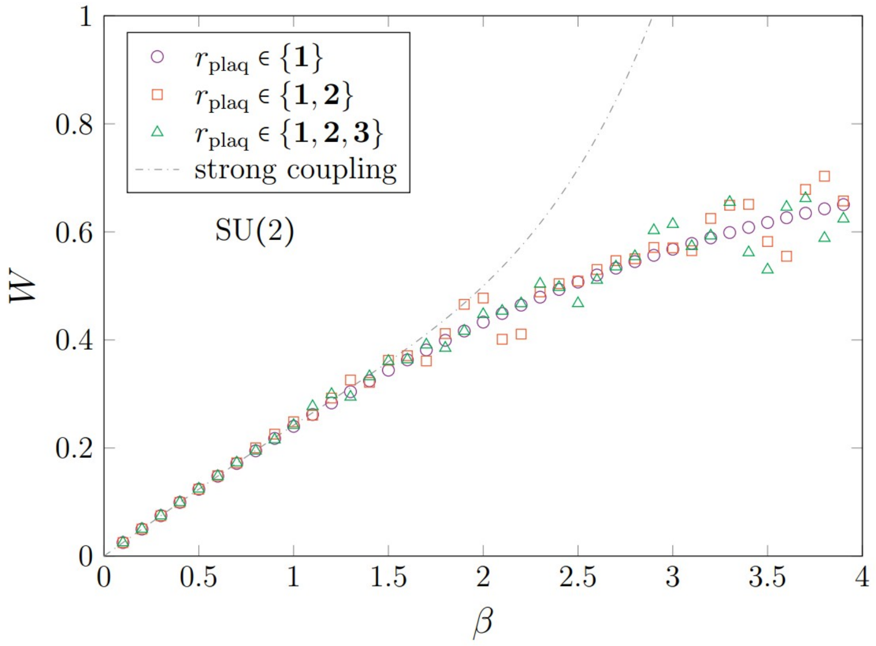
Singular value spectrum of the initial tensor do not have large degeneracy

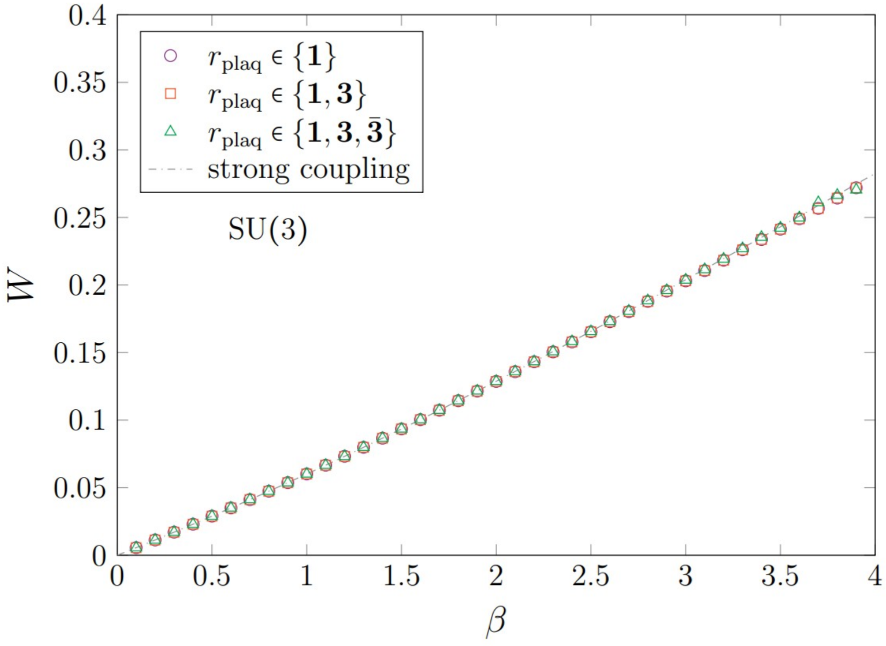
Result: average plaquette @ zero temperature

pure 2+1D SU(2) and SU(3) gauge theory

ATRG; $V = 16^3$; $D_{cut} = 16$

Average plaquette – consistent with strong coupling expansion

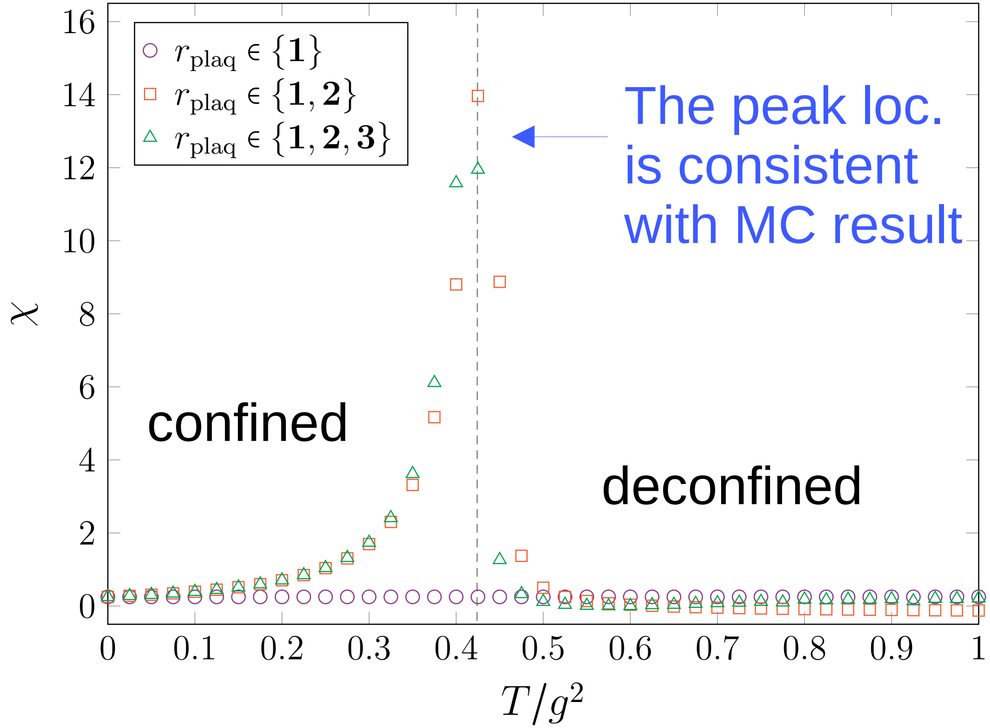


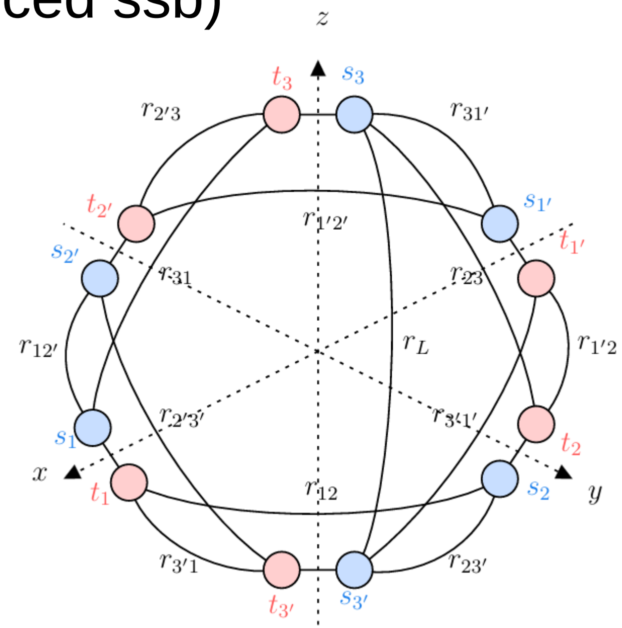


Result: deconfinement @ finite temperature

TRG; V = 1x1024^2; D_{cut} = 64

Polyakov loop susceptibility (with induced ssb)





Armillary sphere with polyakov loop

Summary

- TRG is a promising methods for studying lattice theories
- We address 2 challenging aspects toward lattice QCD
  - Multiple fermion flavors can be handled with Grassmann Tensors with multi-layer construction
  - Degeneracy in non-abelian tensor network can be eliminated with the armillary sphere technique

Future prospects

- Can we reduce the tensor network without character expansion? (Some variation of Gilt-TNR?) [Hauru,Delcamp,Mizera; 2017]
- Armillary sphere method with matter fields?
- More in-depth analysis (physical interpretation?)
- 4D gauge theory + theta term
- Etc.