Scattering phase shift with the tensor renormalization group method

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TENSOR NETWORK 2024

Hadron spectroscopy and scattering phase shift

Hadron spectrum:

Energy gap $\omega_{n,q} = E_{n,q} - E_{\Omega}$ with some quantum number (n, q)

(ex: J^{PC}, flavor, ...)

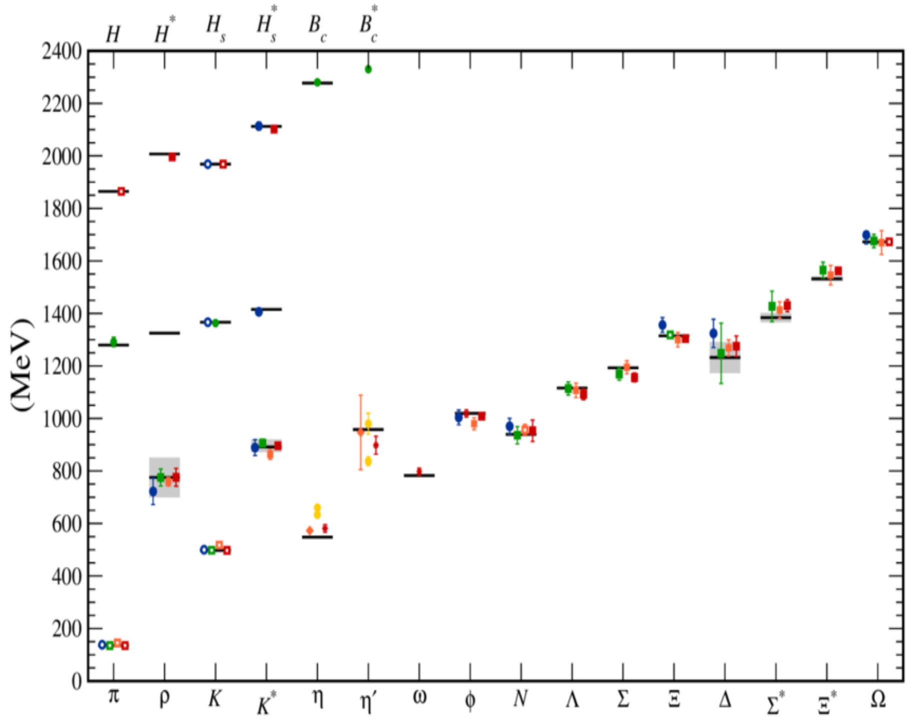
$$\hat{H}_{QCD}|n, q\rangle = E_{n,q}|n, q\rangle$$

Some hadron are resonance ➡ Analysed by phase shift

Lüscher's formula: $e^{i2\delta(k)} = e^{-ikL}$ [Lüscher, 1986]

Phase shift

Relative momentum of **2-particle state energy** with total momentum $P = 0$



2013 snowmass report

Computational method for spectroscopy and phase shift

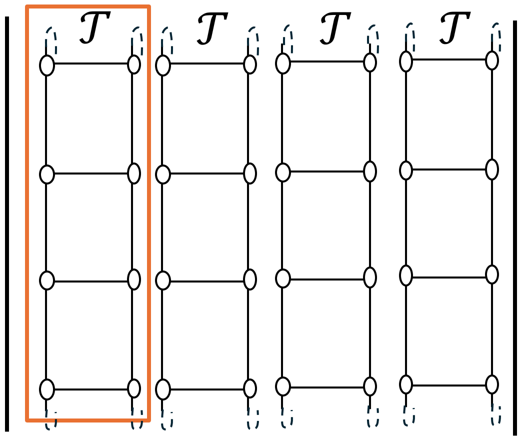
- Monte carlo

$$\lim_{T\to\infty}\langle\hat{\mathcal{O}}_q^{\dagger}(\tau)\hat{\mathcal{O}}_q(0)\rangle = \sum_{n=0}^{\infty}\left|\langle n, q|\hat{\mathcal{O}}_q(0)|\Omega\rangle\right|^2 e^{-\tau(E_{n,q}-E_{\Omega})}$$

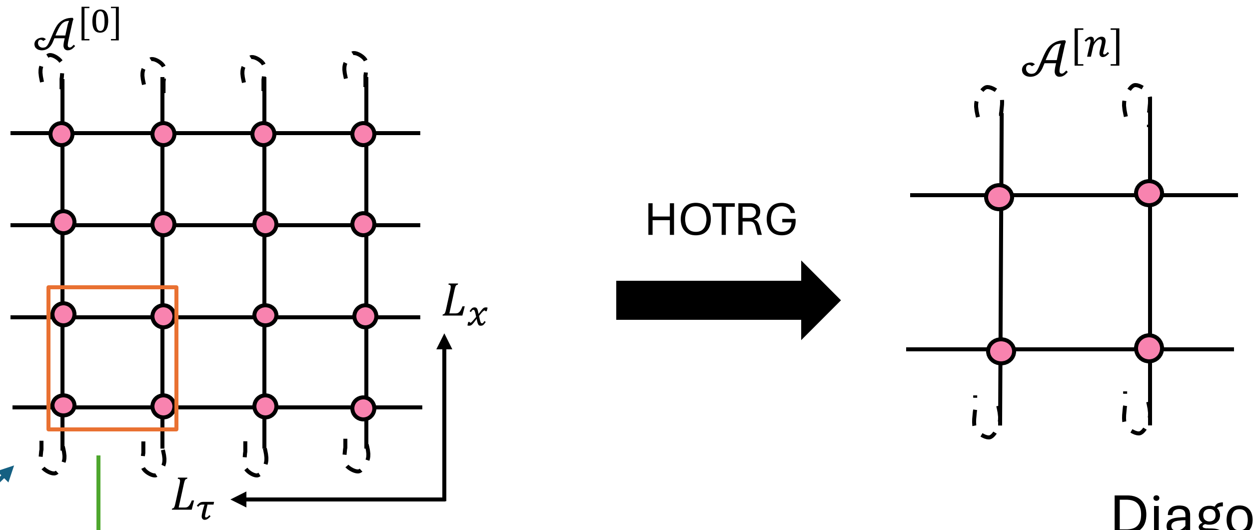
➡ Need large time extend
Large statistics

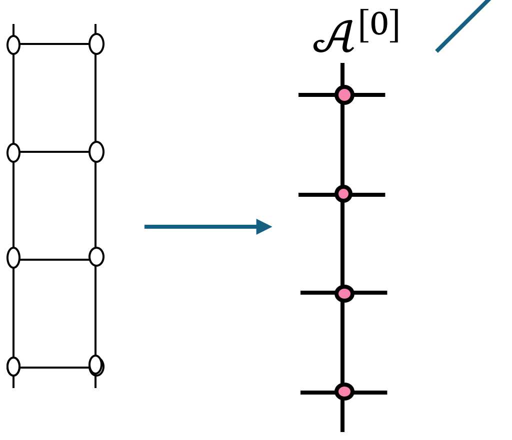
- Transfer matrix and tensor network

Computing the energy spectrum using transfer matrix and tensor network

$Z = \mathrm{Tr}$ 

$$\mathcal{T} = U\lambda U^\dagger = Ue^{-E}U^\dagger$$

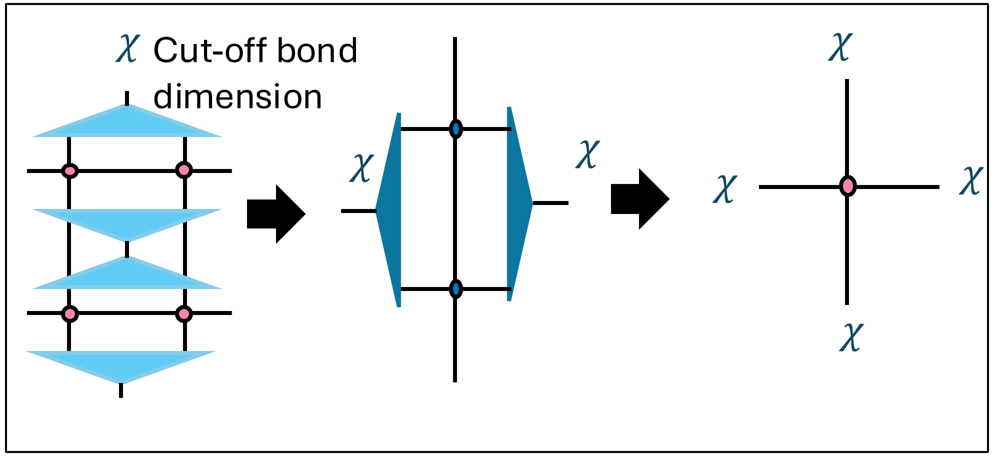




$\mathcal{T} = U\lambda U^\dagger$ $\mathcal{A} = W\lambda W^\dagger$

$\mathbf{dim}\, \mathcal{T} = 2^{L_x}$

Coarse graining
by HOTRG



[Z. Y. Xie, PRB 86 ,2012]

Diagonalize

$$\mathcal{A}^{[n]} = W^{[n]}\lambda^{[n]}W^{[n]\dagger}$$

Approximate eigenvalues of $\mathcal{A}^{L_\tau}$

$$\omega_a^{[n]} = \frac{1}{L_\tau}\log\frac{\lambda_0^{[n]}}{\lambda_a^{[0]}} \approx \omega_a^{[\mathrm{exact}]}$$

Approximation of $\omega_a^{[\mathrm{exact}]} = E_a - E_0$

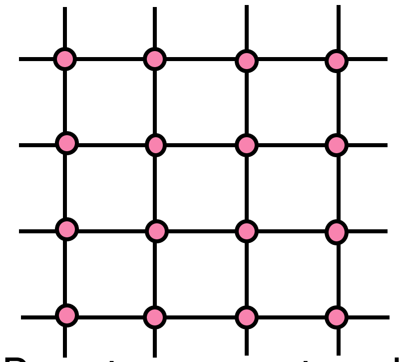
$a = 1,2,3,\ldots,\chi^2$

Identification of quantum number by tensor network

Using the symmetry of the system, matrix elements $\langle\Omega|\hat{\mathcal{O}}|a\rangle$ can be used for

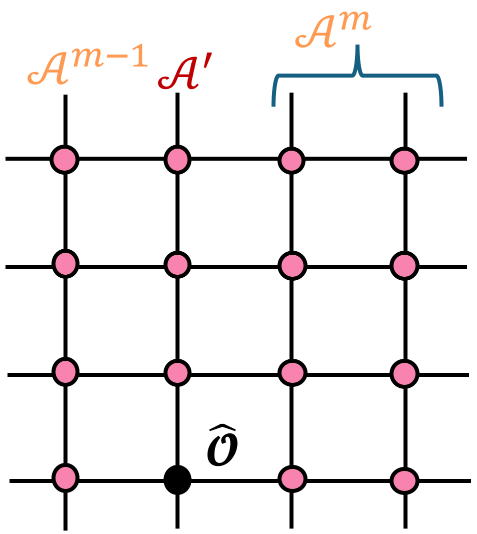
✓ Identification of **quantum number** → if $\langle \mathrm{b}|\hat{\mathcal{O}}|a\rangle \neq 0$ then $q_{\mathrm{b}} q_{\hat{\mathcal{O}}} q_a = 1$ (for discrete symmetry)

$$\langle \boldsymbol{b}|\widehat{\boldsymbol{\mathcal{O}}}|\boldsymbol{a}\rangle = \left(\lambda^{-(m-\frac{1}{2})} W^{\dagger} \mathcal{A}^{m-1} \mathcal{A}' \mathcal{A}^{m} W \lambda^{-(m+\frac{1}{2})}\right)_{ba}$$



$\mathcal{A} = W\lambda W^{\dagger}$

Pure tensor network

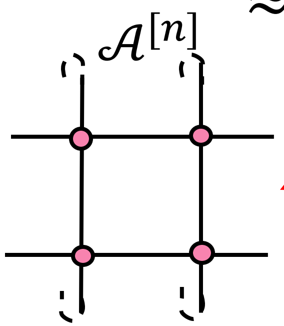


$m = \frac{L_\tau}{2}$

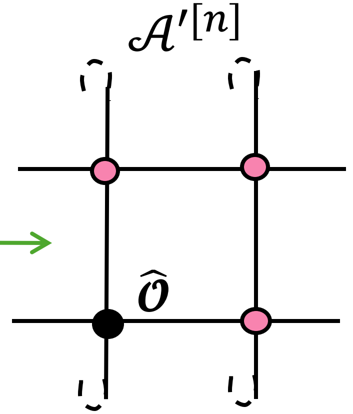
Impurity tensor network for single operator $\widehat{\boldsymbol{\mathcal{O}}}$

After Coarse graining by HOTRG

$$\approx \left(\left(\lambda^{[n]}\right)^{-\frac{1}{L_\tau}(m-\frac{1}{2})} W^{[n]\dagger} \mathcal{A}'^{[n]} W^{[n]} \left(\lambda^{[n]}\right)^{-\frac{1}{L_\tau}(m+\frac{1}{2})} \right)_{ba} = \boldsymbol{B}^{[\mathbf{hotrg}]}_{\boldsymbol{ba}}$$



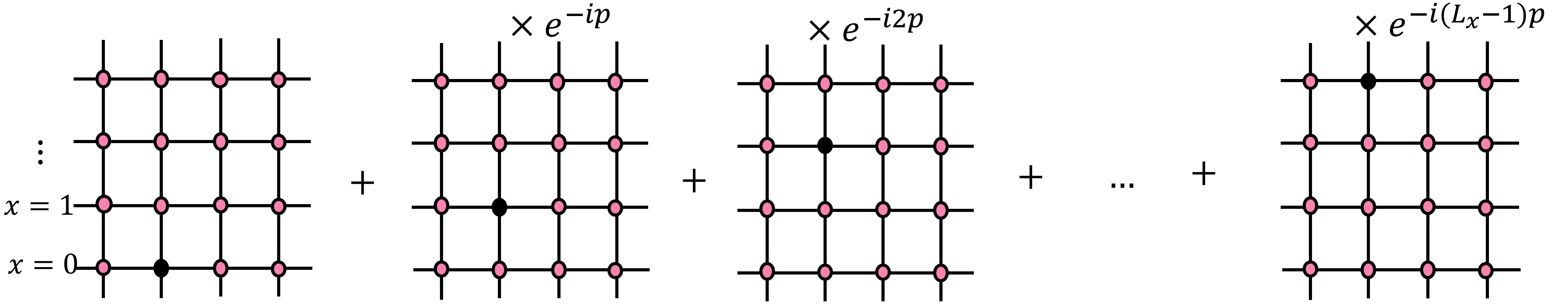
$\mathcal{A}^{[n]} = W^{[n]}\lambda^{[n]}W^{[n]\dagger}$



Identification of momentum by tensor network

✓ Identification of **momentum** → if $\langle\Omega|\hat{\mathcal{O}}(p)|a\rangle \neq 0$ then p is momentum of state $|a\rangle$

$$\langle\Omega|\hat{\mathcal{O}}\,(p)|a\rangle = \langle\Omega|\frac{1}{L_x}\sum_{x=0}^{L_x-1}\hat{\mathcal{O}}(x)e^{-ipx}\;|a\rangle \approx B_{0a}(p)^{[\mathrm{hotrg}]}$$

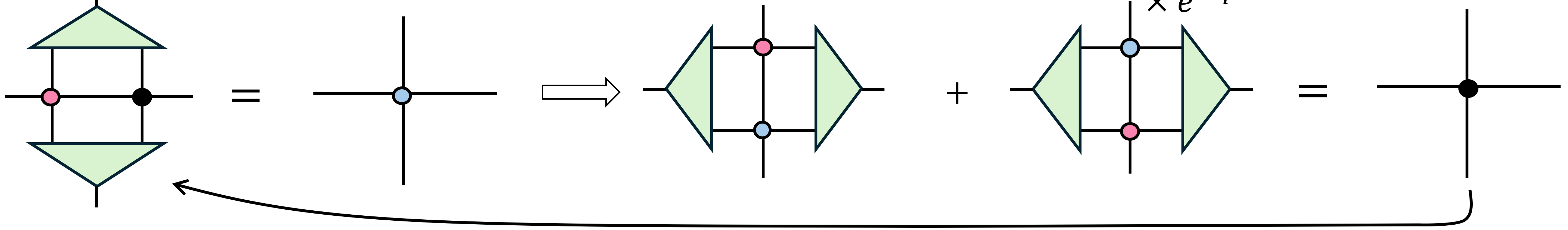


$\langle\Omega|\hat{\mathcal{O}}(0)|a\rangle +$ $\langle\Omega|\hat{\mathcal{O}}(1)|a\rangle e^{-ip} +$ $\langle\Omega|\hat{\mathcal{O}}(2)|a\rangle e^{-i2p} +$ $+\,\langle\Omega|\hat{\mathcal{O}}(L_x-1)|a\rangle e^{-i(L_x-1)p}$

Cost: $O(L_x\,\chi^7)$

Coarse-graining $\langle\Omega|\hat{\mathcal{O}}\,(p)|a\rangle$,
Cost: $O(\log L_x\,\chi^7)$

[S. Morita, N. Kawashima, 2019]



Energy spectrum and quantum number of (1+1)d Ising model

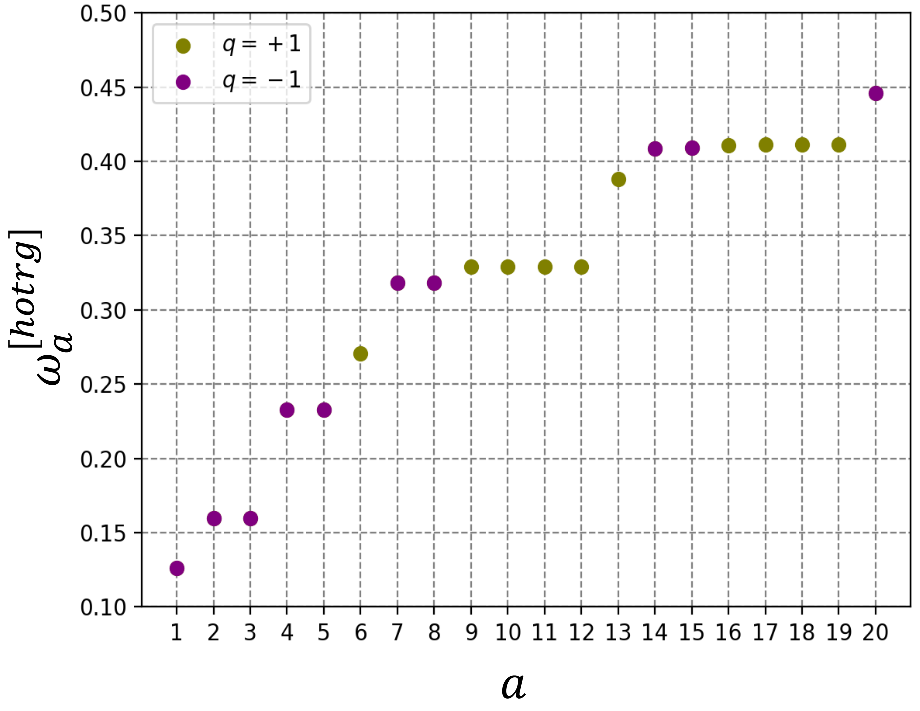
Classification of quantum number:

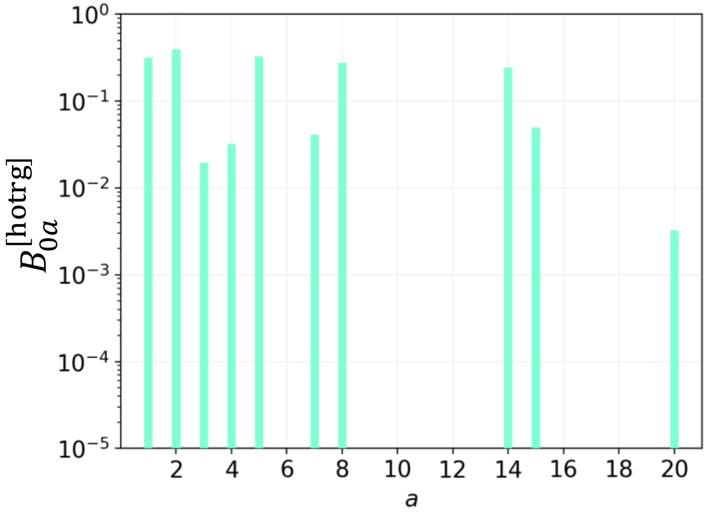
$$|\langle \Omega | s(0) | a \rangle| \approx |B_{0a}^{[\mathrm{hotrg}]}|$$

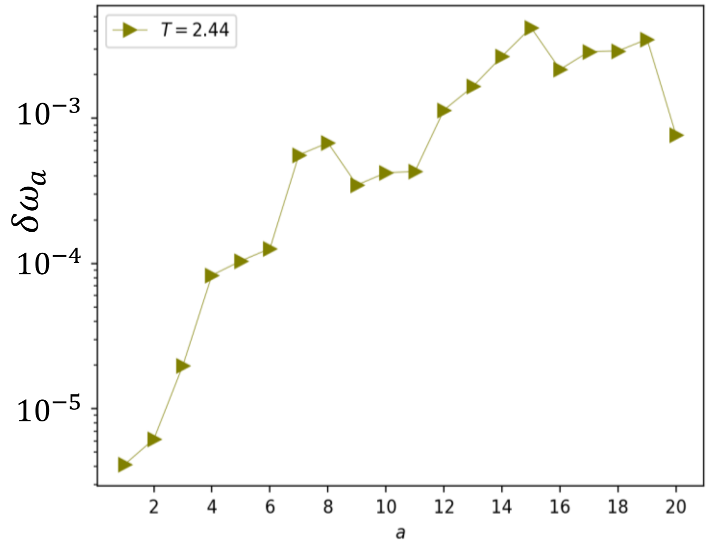
Single spin operator , $q_s = -1$

$$\left| B_{0a}^{[\mathrm{hotrg}]} \right| \neq 0 \Rightarrow \underset{+1\;-1}{q_\Omega q_s} q_a = 1, \Leftrightarrow \boldsymbol{q_a = -1}$$

$T = 2.44, L_x = 64, \chi = 80$







$$\delta\omega_a = \frac{\left|\omega_a^{[hotrg]} - \omega_a^{[\mathrm{exact}]}\right|}{|\omega_a^{[\mathrm{exact}]}|}$$

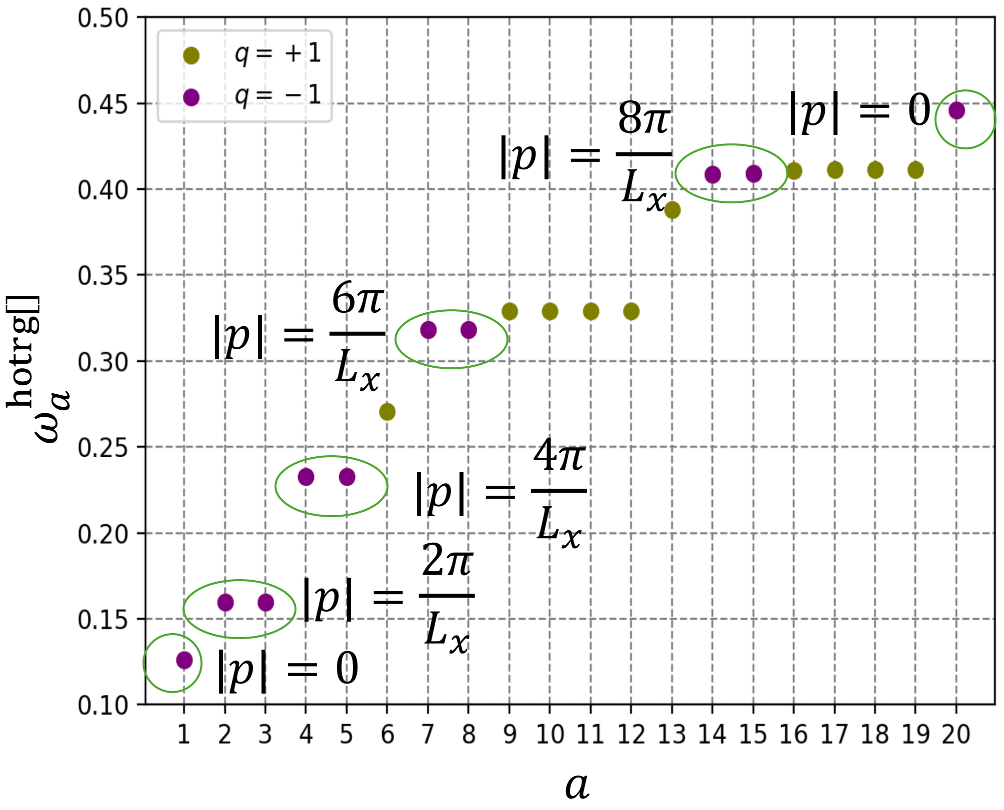
[Kaufman, Phys. Rev. 76, (1949)]

Momentum of 1-particle state of (1+1)d Ising model

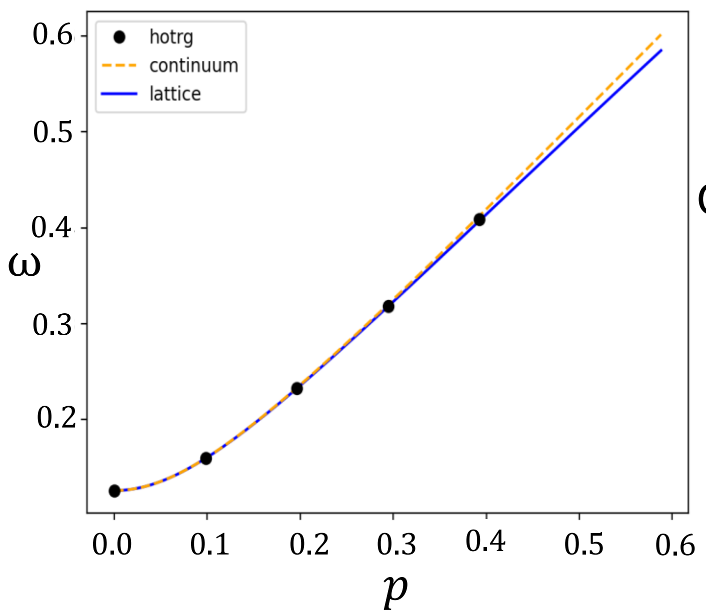
Identification of momentum of 1- particle state ($q = -1$ sector)

if $|\langle\Omega| \frac{1}{L_x} \sum_{x=0}^{L_x-1} s(x) e^{-ipx} |a\rangle| \neq 0$, p is momentum of $|a\rangle$

Coarse grained by HOTRG



Dispersion relation



Continuum:

$\omega = \sqrt{m^2 + p^2}$

Lattice:

$\omega = \cosh^{-1}(1 - \cos p + \cosh m)$

Scattering phase for shift (1+1)d Ising

Operator for identification 2- particle state with total momentum $\boldsymbol{P} = \mathbf{0}$

relative momentum

$$|\langle\Omega|\hat{\mathcal{O}}_2(P,p)|a\rangle| = |\langle\Omega|\frac{1}{L_x^2}\sum_{x,y=0}^{L_x-1} s(x)s(y)e^{-ip_1x}e^{-ip_2y}|a\rangle$$

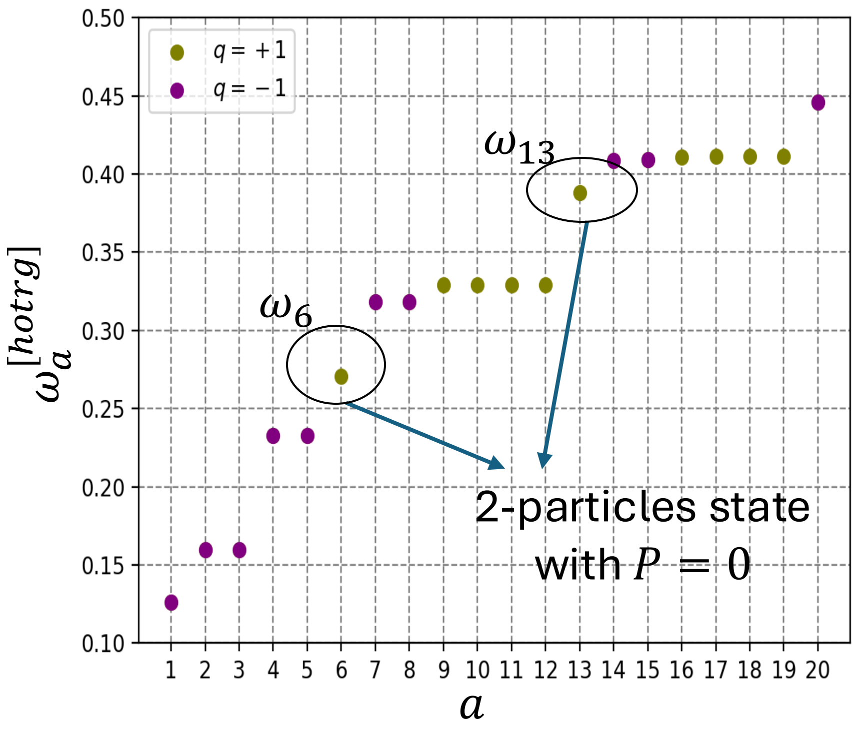
Coarse grained by HOTRG

- $P = p_1 + p_2$
- $p = \frac{p_1 - p_2}{2}$

2-particle state energy with $\boldsymbol{P} = \mathbf{0}$ ($\boldsymbol{T} = \mathbf{2.44}$, $\boldsymbol{\chi} = \mathbf{80}$)

			$\boldsymbol{P = 0, p = 0}$	$\boldsymbol{P \neq 0}$
L_x	a	$\omega_a^{[\text{hotrg}]}$	$\langle\Omega\|\mathcal{O}_2(0,0)\|a\rangle$	$\langle\Omega\|\mathcal{O}_2(2\pi/L_x,\pi/L_x)\|a\rangle$
8	4	0.814585	0.37740	$< 10^{-15}$
	19	2.133922	0.07730	$< 10^{-12}$
16	4	0.465348	0.31004	$< 10^{-15}$
	18	1.171480	0.06904	$< 10^{-12}$
32	4	0.319553	0.21122	$< 10^{-14}$
	14	0.636356	0.04705	$< 10^{-10}$
64	6	0.270836	0.12007	$< 10^{-14}$
	13	0.387849	0.03007	$< 10^{-9}$

$T = 2.44, L_x = 64, \chi = 80$



Scattering phase shift for (1 + 1)d Ising

Relative momentum

Lüscher's formula,

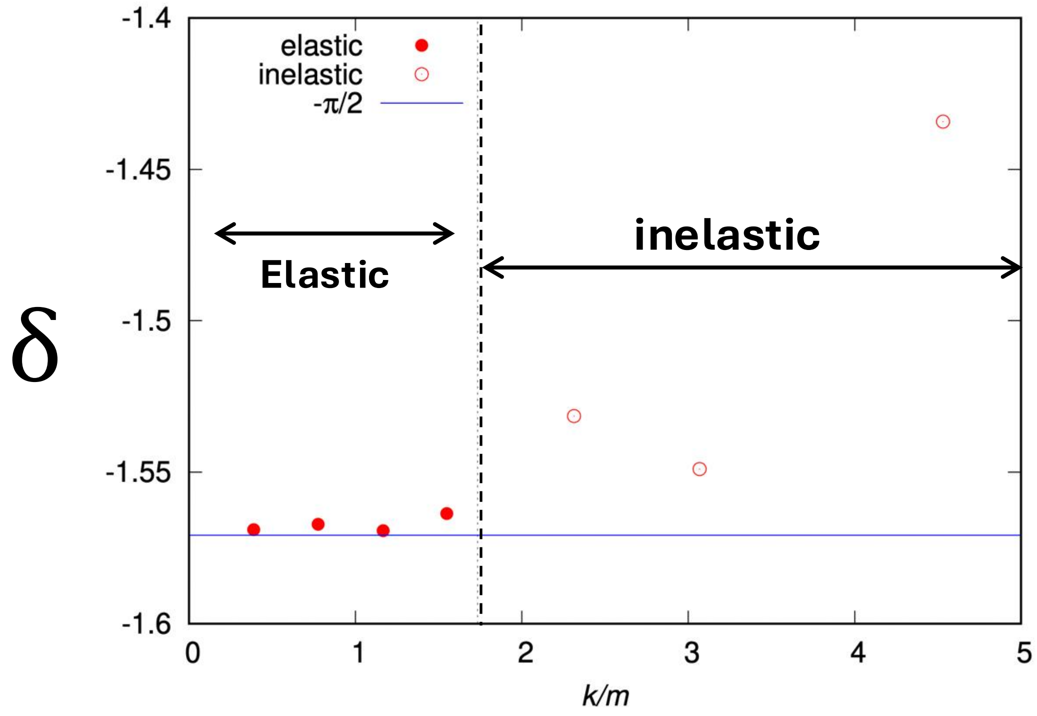
$$e^{i2\delta(k)} = e^{-ikL_x}$$

Phase shift

L_x	a	$\omega_a^{[\mathrm{hotrg}]}$
8	4	0.814585
	19	2.133922
16	4	0.465348
	18	1.171480
32	4	0.319553
	14	0.636356
64	6	0.270836
	13	0.387849

Elastic region

$2m \leq \omega < 4m$



$$\omega = 2\sqrt{k^2 + m^2}$$

Relative momentum

infinite volume limit
exact rest mass
$m = 0.12621870$

$\delta_{\mathrm{ising}} = -\frac{\pi}{2}$ [C. R. Gattringer,1993]

Summary and outlook

- By using our scheme, the energy spectrum is obtained from eigenvalues of transfer matrix which is approximated by tensor network
- The the quantum number is judged from the matrix elements of a proper operator
- The momentum of one-particle state energy can be identified
- The two-particle state energy with total momentum zero can be identified
- Using Lüscher's formula, the scattering phase shift of 2d Ising model is obtained from two-particle state energy whose total momentum is zero
- outlook: application to other lattice models, phase shift from moving frame, etc.

Appendix

Error of relative momentum and phase shift

L_x	a	$k^{[\text{hotrg}]}$	$k^{[\text{exact}]}$	δk
8	0	0.387241735	0.387241735	0
	1	1.059468963	1.059468963	0
16	0	0.195463359	0.195463359	0
	1	0.571978995	0.57197883	3×10^{-7}
32	0	0.097966346	0.097966084	3×10^{-6}
	1	0.292071859	0.292034069	0.000129
64	0	0.049059827	0.049012477	0.000966
	1	0.147225924	0.146802317	0.002885

Relative error of the relative momentum $k^{[\text{hotrg}]}$

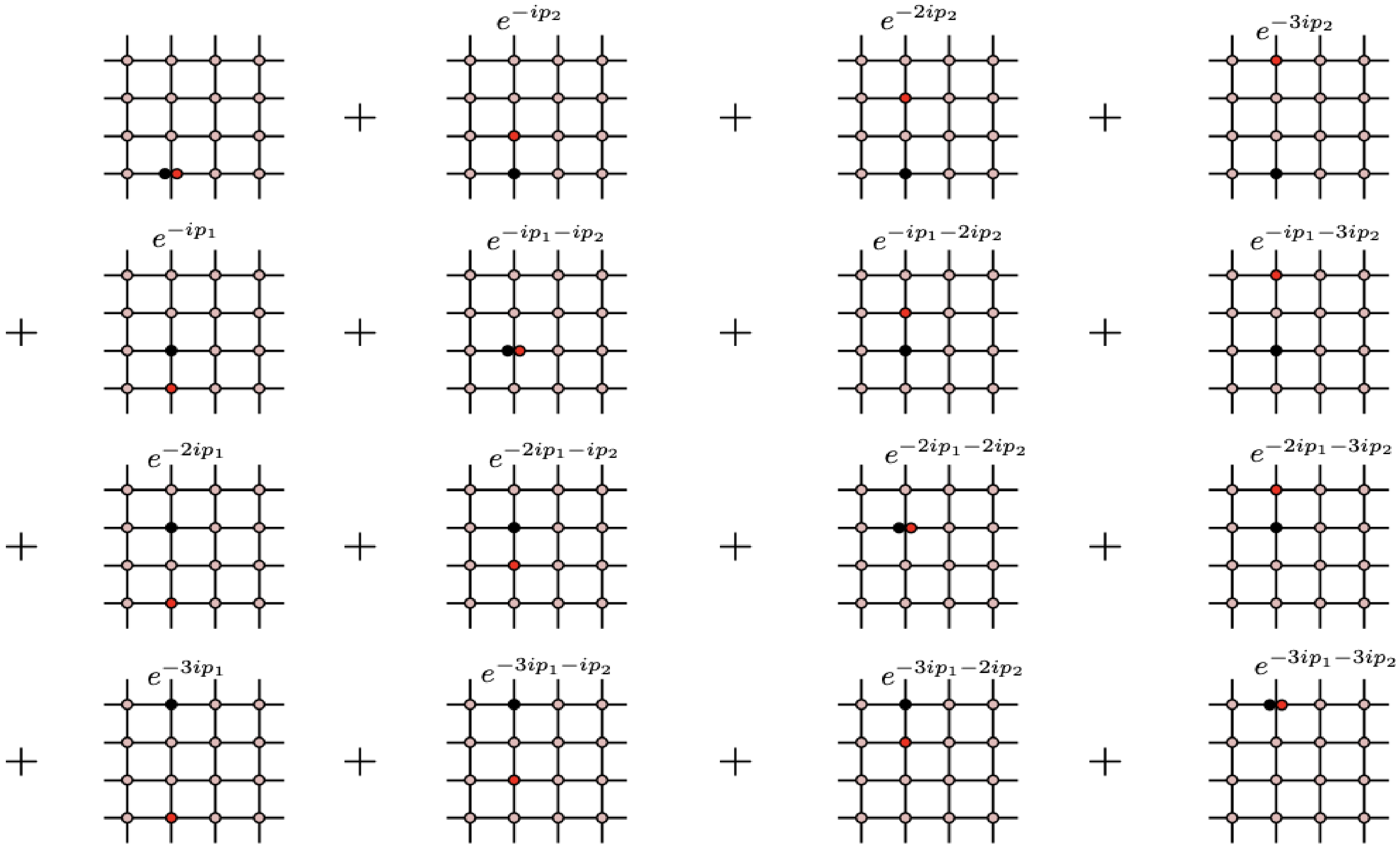
L_x	a	$\delta(k)^{[\text{hotrg}]}$	$\delta(k)^{[\text{exact}]}$	$\Delta\delta(k)$
8	0	-1.5489669	-1.5489669	0
	1	-1.0962832	-1.0962832	0
16	0	-1.5637069	-1.5637069	0
	1	-1.4342393	-1.4342393	9×10^{-7}
32	0	-1.5674615	-1.5674573	3×10^{-6}
	1	-1.5315571	-1.5315571	0.000394
64	0	-1.5699145	-1.5683993	0.000966
	1	-1.5696369	-1.5560815	0.008711

Relative error of the phase shift $\delta(k)^{[\text{hotrg}]}$

Impurity TN to compute Scattering phase shift (1+1)d Ising

Operator for identification 2- particle state with total momentum $P = 0$, $L_x = 4$

$$|\langle\Omega|\widehat{\mathcal{O}}_2(P,p)|a\rangle| = |\langle\Omega|\frac{1}{L_x^2}\sum_{x,y=0}^{L_x-1} s(x)s(y)e^{-ip_1x}e^{-ip_2y}\,|a\rangle$$



Tensor Network Representation for Momentum of 2 −particle state

Momentum operator for 2-particle state

$$|\langle\Omega|\hat{\mathcal{O}}_2(P,p)\;|a\rangle| = |\langle\Omega|\frac{1}{L_x^2}\sum_{x,y=0}^{L_x-1} s(x)s(y)e^{-ip_1x}e^{-ip_2y}\;|a\rangle|$$

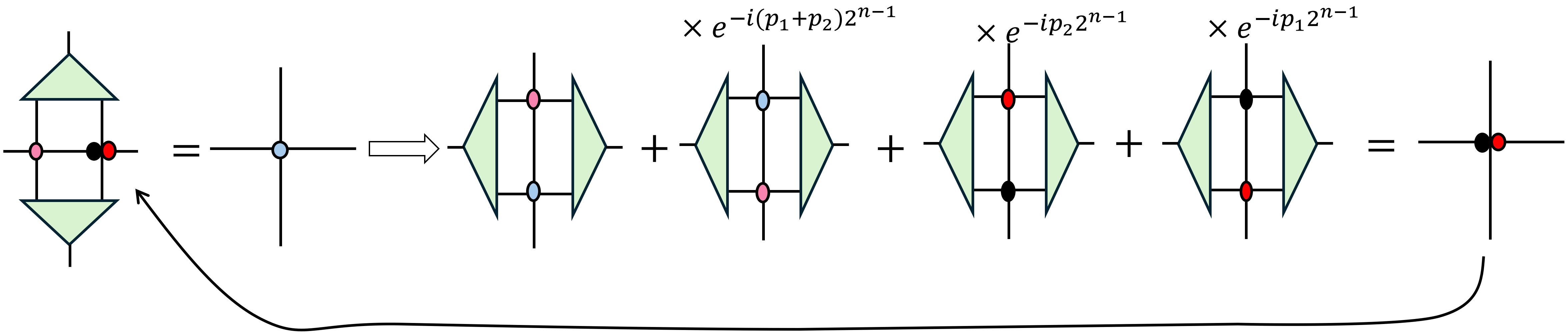
$p_1 = \frac{2\pi n_1}{L_x}$ $\quad$ $p_2 = \frac{2\pi n_2}{L_x}$

$P = p_1 + p_2$ total momentum

$p = \frac{p_1 - p_2}{2}$ relative momentum

For a given P, if $|\langle\Omega|\hat{\mathcal{O}}_2(\mathrm{P},\mathrm{p})\;|\mathrm{a}\rangle| \neq 0 \implies$ total momentum of state $|\mathrm{a}\rangle$ is P

We coarse grain L_x^2 tensor networks to compute $\langle\Omega|\hat{\mathcal{O}}_2(P,p)\;|a\rangle$ by using:



Cost: $O(L_x \log L_x\, \chi^7)$

n is number of coarse-graining step

Tensor Network Representation for $\langle b|\hat{\mathcal{O}}_q|a\rangle$

$\mathcal{T} = U\lambda U^\dagger$

$$\langle b|\hat{\mathcal{O}}_q|a\rangle = \left(U^\dagger\, \hat{\mathcal{O}}_q U\right)_{ba}$$

$m = L_\tau/2$ ($\hat{\mathcal{O}}_q$ is in the middle of square lattice)

$$= \left(U^\dagger \mathcal{T}^{-m}\mathcal{T}^{m} \hat{\mathcal{O}}_q \mathcal{T}^{(m+1)}\mathcal{T}^{-(m+1)} U\right)_{ba}$$

Using $\mathcal{T}\,\mathcal{T}^{-1} = I$

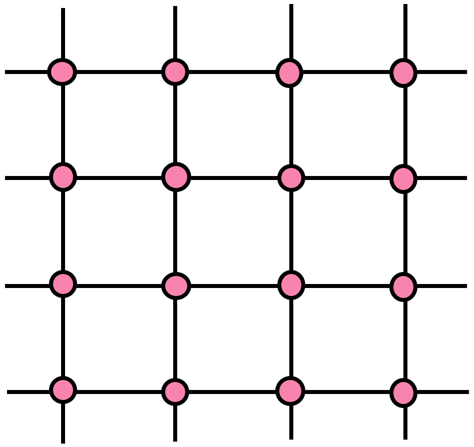
$U\lambda^{-m}U^\dagger$ $(YY^\dagger)^m$ $(YY^\dagger)^{m+1}$ $U\lambda^{-(m+1)}U^\dagger$

$$= \left(\lambda^{-m}U^\dagger Y\left(Y^\dagger Y\right)^{m-1} Y^\dagger \hat{\mathcal{O}}_q Y \left(Y^\dagger Y\right)^{m} Y^\dagger U\lambda^{-(m+1)}\right)_{ba}$$

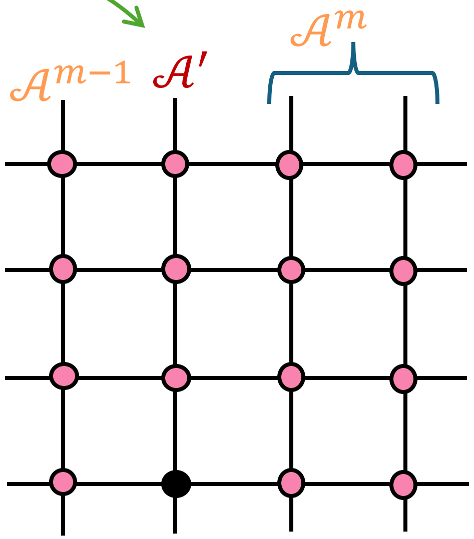
$U\sqrt{\lambda}W^\dagger$ $\mathcal{A}$ $\mathcal{A}'$ $\mathcal{A}$ $U^\dagger\sqrt{\lambda}W$

$$= \left(\lambda^{-(m-\frac{1}{2})} W^\dagger \mathcal{A}^{m-1}\mathcal{A}'\mathcal{A}^{m} W \lambda^{-(m+\frac{1}{2})}\right)_{ba}$$

$\mathcal{A} = W\lambda W^\dagger$



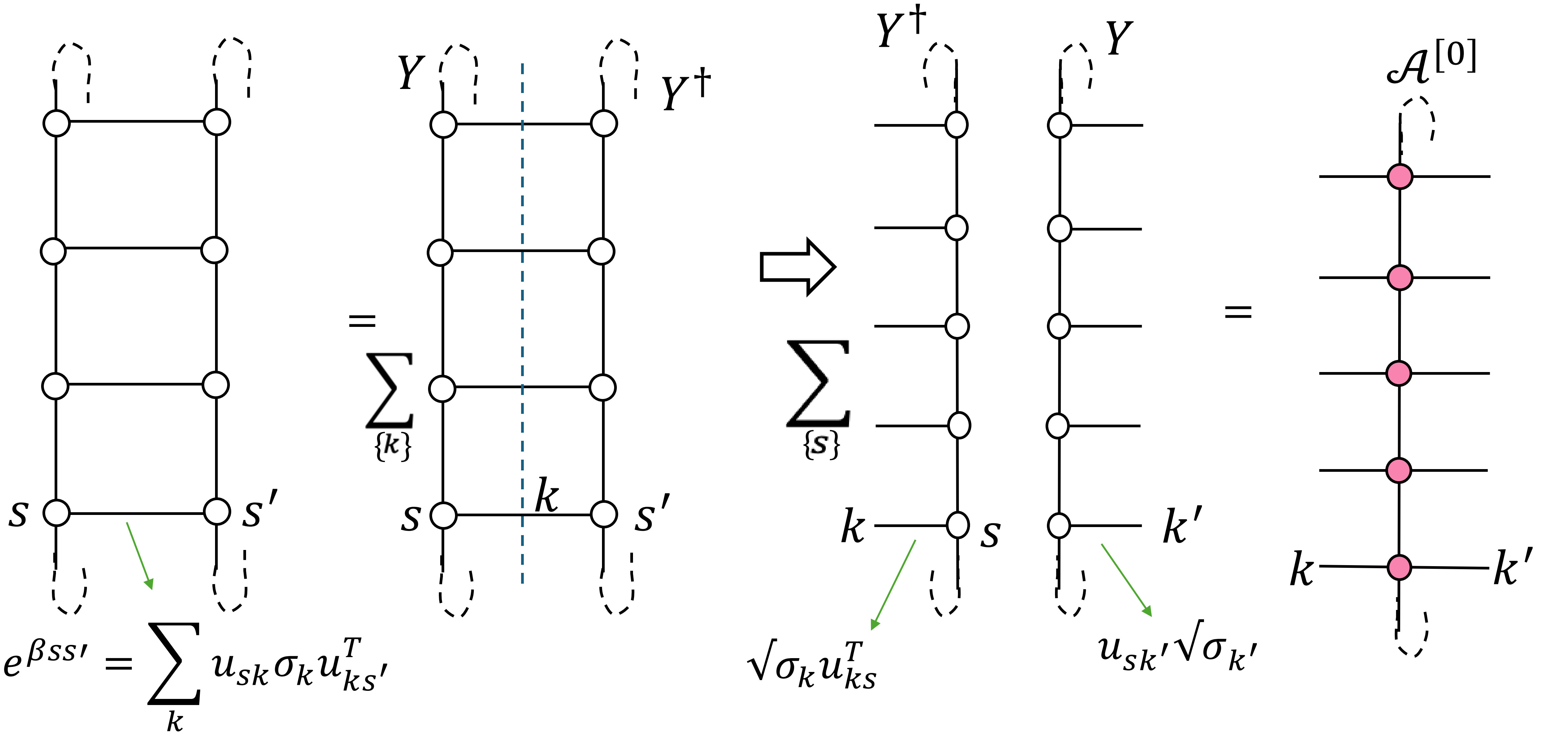
All terms are obtained from Tensor Network



Impurity Tensor Network

Transfer Matrix Spectrum by Tensor Network

$$Z = Tr[\mathcal{T}\,\mathcal{T}\,\dots] \;= Tr[{\color{red}Y}Y^{\dagger}YY^{\dagger}\dots YY^{\dagger}] \;= Tr[Y^{\dagger}\,YY^{\dagger}Y\dots Y^{\dagger}{\color{red}Y}] \;= Tr[\mathcal{A}^{[0]}\mathcal{A}^{[0]}\dots]$$



Identification of quantum numbers

System with Discrete Symmetry

Ex: (1+1)d Ising Model, Sym over Z_2, $q = \pm 1$

Let $\widehat{D}$ be a discrete transformation operator.

Discrete transformation of operator $\hat{X}$ is

$$\widehat{D}\hat{X}\widehat{D}^{-1} = q_X \hat{X}$$

$$\widehat{D}|a\rangle = q_a|a\rangle$$

$$\begin{aligned}\langle \mathrm{b}|\hat{X}|a\rangle &= \langle \mathrm{b}|\widehat{D}^{-1}\widehat{D}\hat{X}\widehat{D}^{-1}\widehat{D}|a\rangle \\ &= q_b q_X\, q_a \langle b|\hat{X}|a\rangle\end{aligned}$$

This gives us selection rule:

$$\langle \mathbf{b}|\boldsymbol{X}|\boldsymbol{a}\rangle \neq \mathbf{0} \;\Rightarrow \boldsymbol{q_b q_X q_a} = \mathbf{1}$$

q_X Assumed to be known
Choose $\langle b|$ as $\langle \Omega|$ where $q_\Omega = +1$

Then q_a can be identified

Identification of quantum numbers

<u>For system with Continious symmetry</u>

- Let $\widehat{Q}$ be a conserved charge of continious symmetry and $\left[\widehat{Q}\,,\widehat{H}\right] = 0$
- If Quantum number of an operator $\widehat{X}$ is q_X then

$$\left[\widehat{Q}\,,\widehat{X}\,\right] = q_X\widehat{X}$$

Assume $|\Omega\rangle$ has no charge $\widehat{Q}\,|\Omega\rangle = 0$,

$$\hat{Q}\hat{X}\,|\Omega\rangle = q_X\widehat{X}\,|\Omega\rangle$$

For energy eigenstate $|a\rangle, |b\rangle$

$$\langle b|(\hat{Q}\hat{X} - \hat{X}\hat{Q})|a\rangle = \langle b|q_X\widehat{X}\,|a\rangle$$

$$(q_a - q_b - q_X)\langle b|\widehat{X}\,|a\rangle = 0$$

Selection Rule:

If $\left\langle b\middle|\widehat{X}\,\middle|a\right\rangle \neq 0$, then $(q_a - q_b - q_X) = 0$

If $\langle b| = \langle\Omega|$ then $q_a = q_X$