Tensor network approach to studying fractal lattice

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Motivation

- Fractals often appear in nature.
  (self-similarity, non-integer fractal dimensions, ⋯)

- Physical models on fractal spaces.
  (Gefen-Mandelbrot-Aharony (1980, 1982, 1983, 1984))

→ Universality? A field-theoretic description on a fractal?

- Dimensional regularization and quantum gravity models, involve the emergence of non-integer dimensions.

→ Are fractals significant?

- Self-similar repeating structure → Suitable for TRG calculations?

In this talk, I consider the Ising model on a fractal space,
and calculate various physical quantities using TRG to understand
phyisical models on fractals.

Contents

1. What are Fractals

Fractal

- Self-similarity

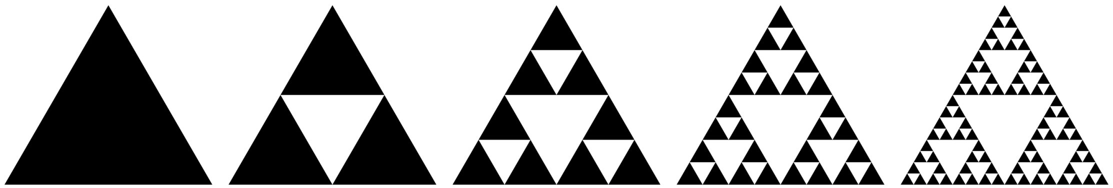
https://minorinosato-togane.com/2016/03/07/8880

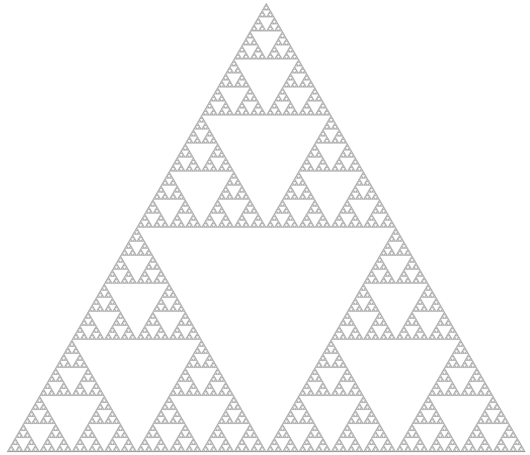
(Examples) the boundaries of clouds, coastlines, mountain surfaces, and so on.

Hausdorff dimension

Normally, when the length is L, the volume in d dimensions becomes $S = L^d$

$$d = \log S / \log L$$



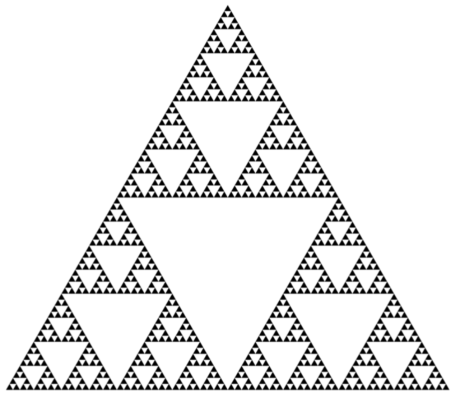


$$d_H = \frac{\log 3}{\log 2} \approx 1.584962$$

Statistical models on Sierpinski gasket/carpet

The Ising model on a Sierpiński gasket

- Finite bond cuts → decomposable into clusters (ramification is finite) → 1D-like
- No phase transition

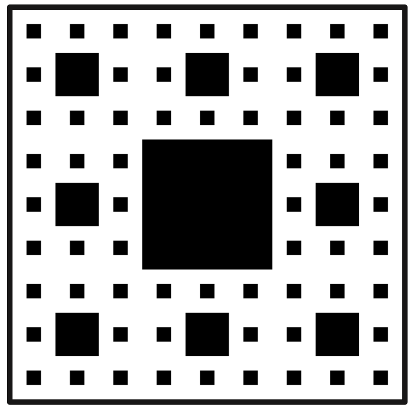


Sierpiński gasket

Sierpinski carpet

- Ramification is infinite.
- Phase transition occurs.

We study the case of Sierpinski carpet using TRG.



Sierpiński carpet

2. TRG on Sierpiński carpet

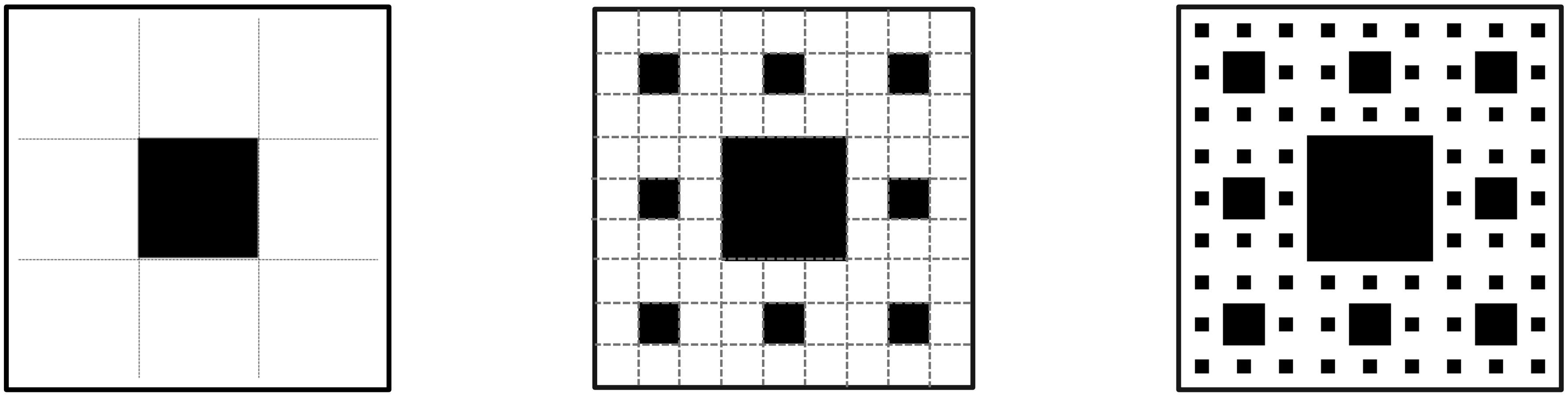
Sierpiński carpet - $S(b,c)$

A single square is divided into b^2 smaller squares, from which c^2 squares are removed.

$$d_{\mathrm{H}} = \log(b^2 - c^2)/\log b$$

例 $\mathrm{SC}(3,1)$

$n=1$ $n=2$ $n=3$



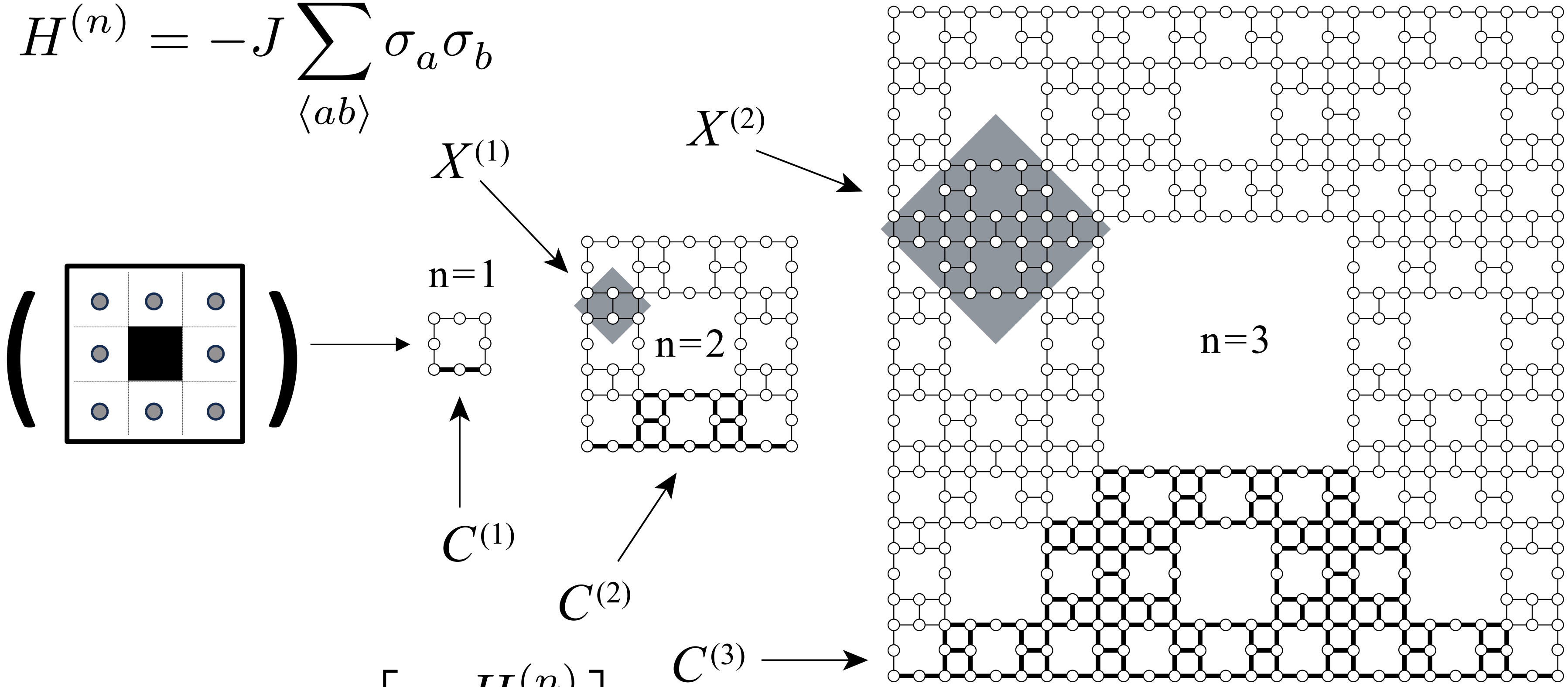
$$d_{\mathrm{H}} = \log_3 8 \approx 1.8927$$

→ Spins are placed on the faces.

The Ising model on the Sierpiński carpet

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)

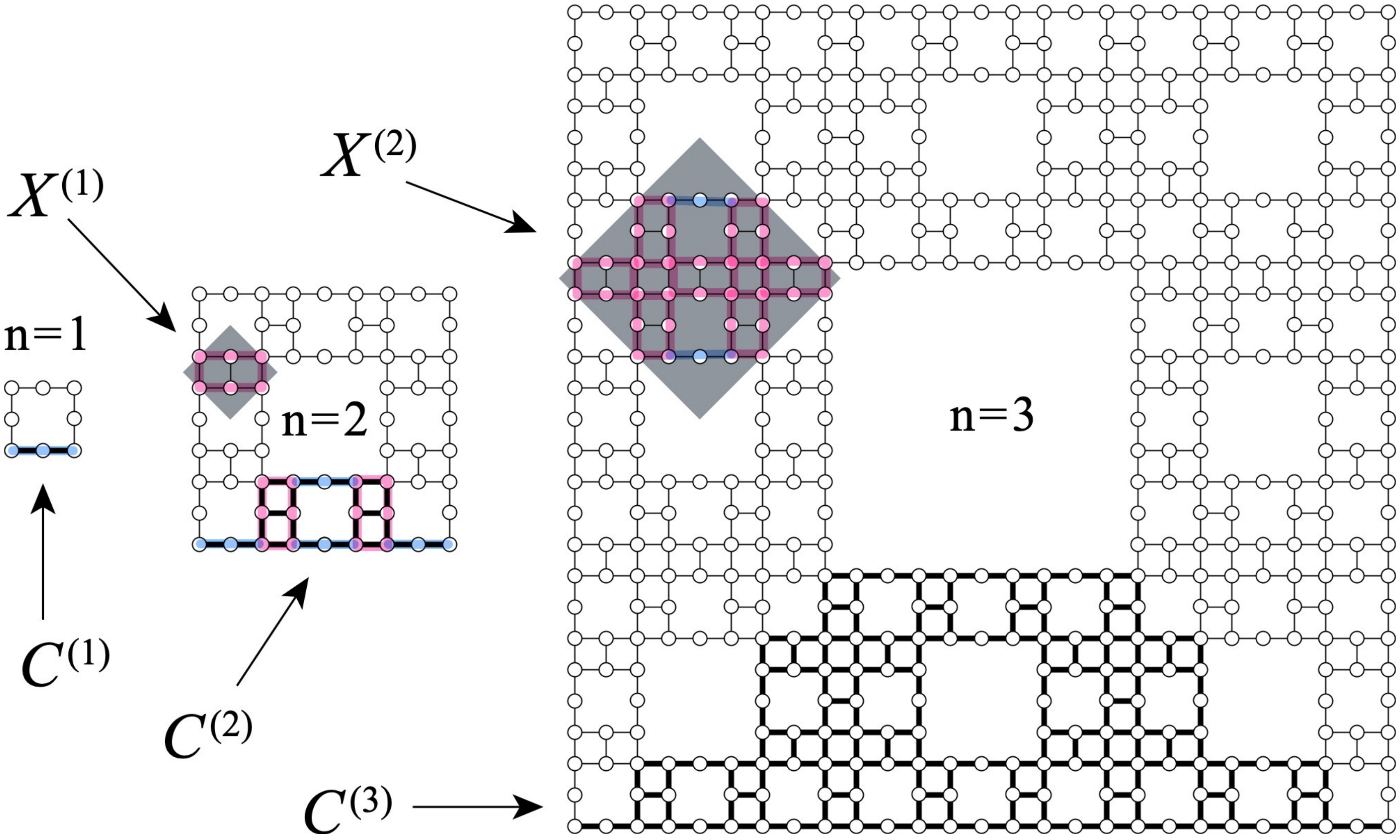
$$H^{(n)} = -J \sum_{\langle ab \rangle} \sigma_a \sigma_b$$



$$Z^{(n)} = \sum \exp\left[-\frac{H^{(n)}}{k_{\mathrm{B}}T}\right]$$

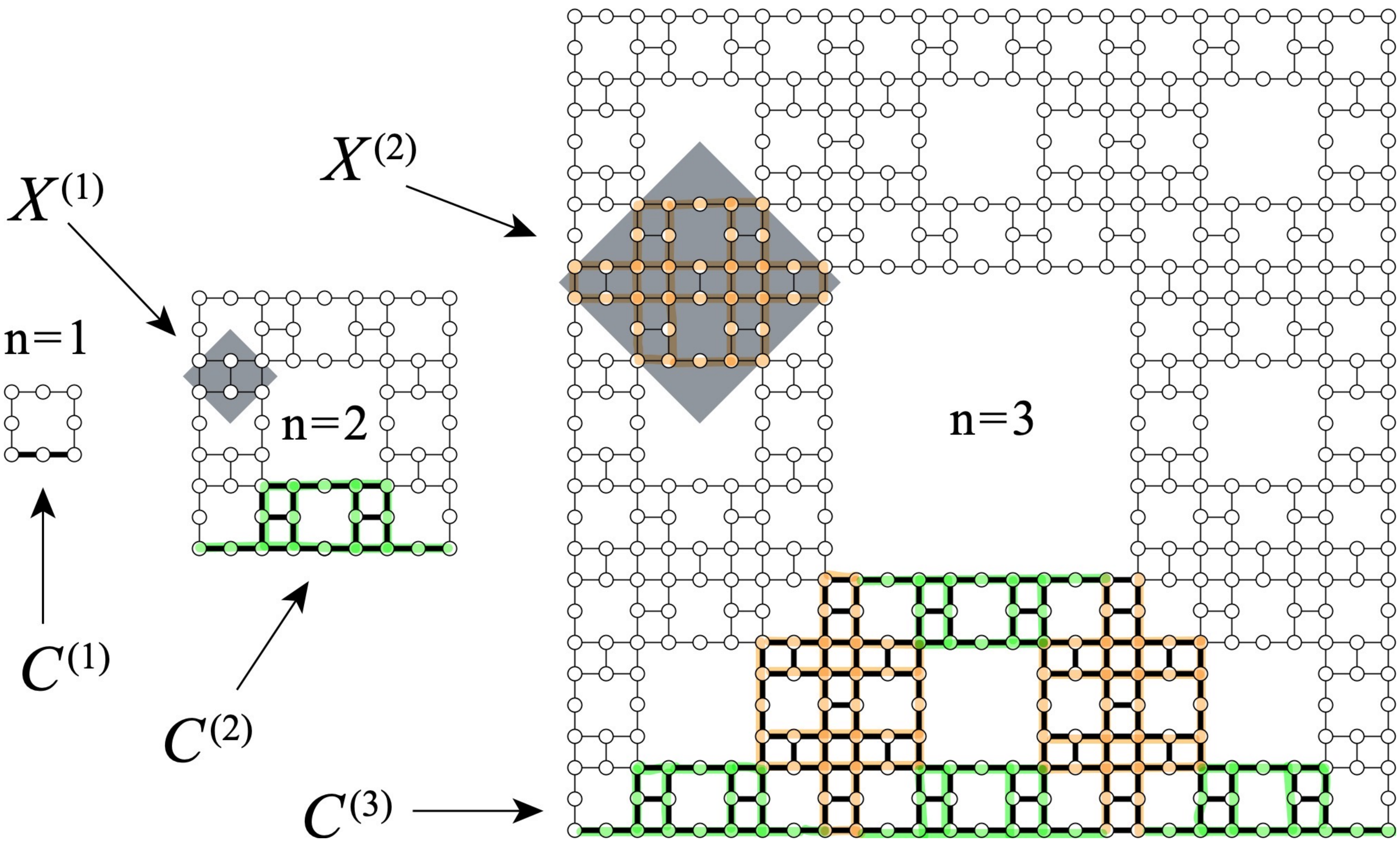
The Ising model on the Sierpiński carpet

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)

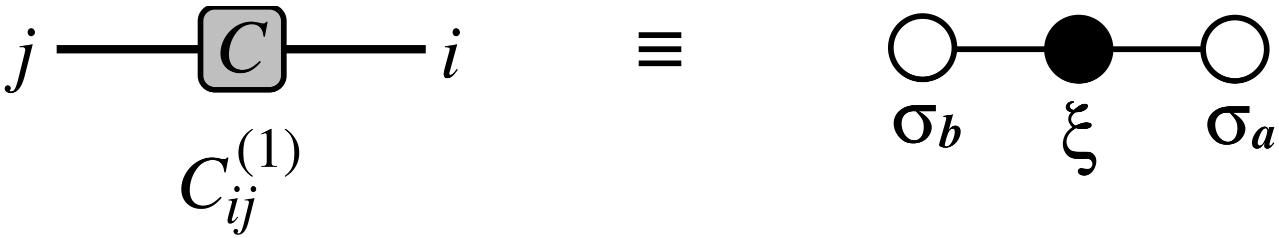


The Ising model on the Sierpiński carpet

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)



Corner Matrix $C^{(1)}$

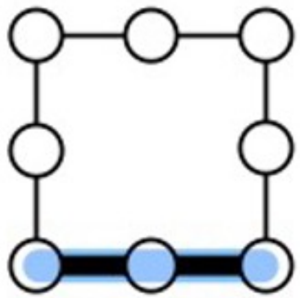


$$C_{ij}^{(1)} = \sum_{\xi=\pm 1} \exp\left[K\xi\left(\sigma_a + \sigma_b\right)\right]$$

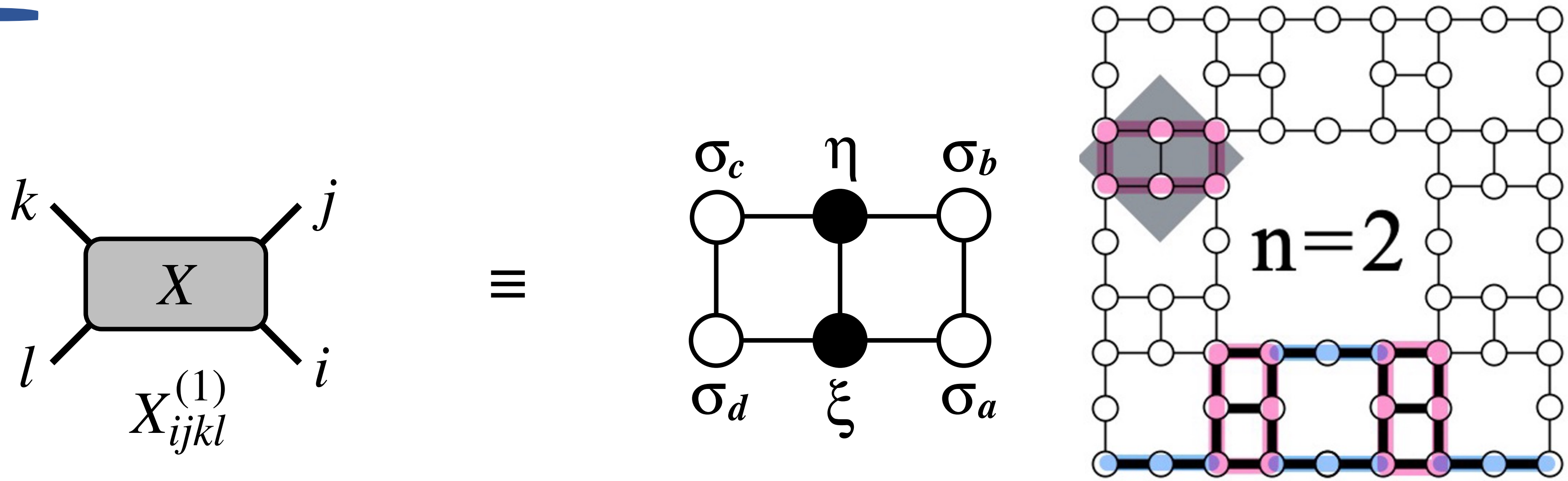
where $\quad i = (\sigma_a + 1)/2,\ j = (\sigma_b + 1)/2$

$n = 1$

$$Z^{(1)} = \sum_{ijkl} C_{ij}^{(1)}\, C_{jk}^{(1)}\, C_{kl}^{(1)}\, C_{li}^{(1)} = \mathrm{Tr}\left[C^{(1)}\right]^4 \quad =$$



$X^{(1)}$



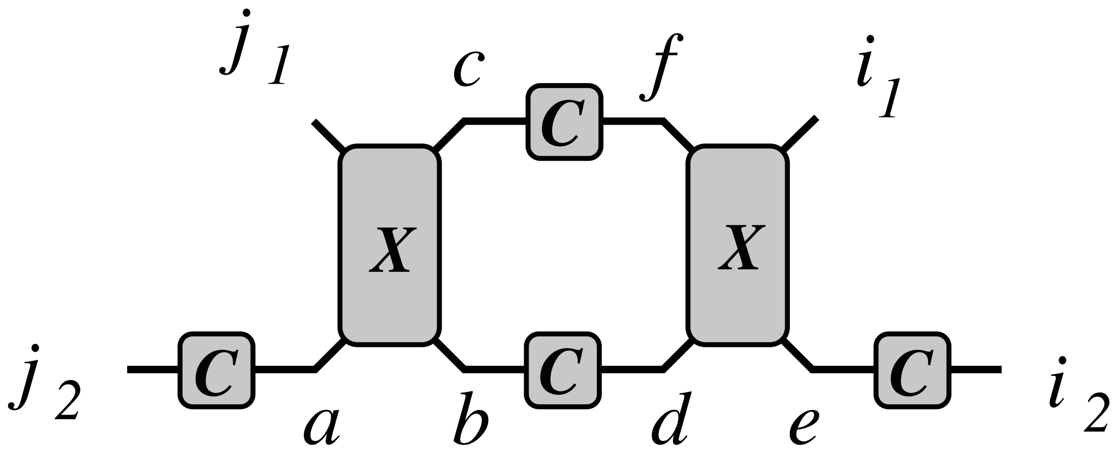
$$X^{(1)}_{ijkl} = \sum_{\xi\eta} \exp\left[K\left(\sigma_a\sigma_b + \sigma_c\sigma_d + \xi\eta\right)\right]$$

$$\times \exp\left[K\xi\left(\sigma_a + \sigma_d\right) + K\eta\left(\sigma_b + \sigma_c\right)\right]$$

New C

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)

C^{new} =



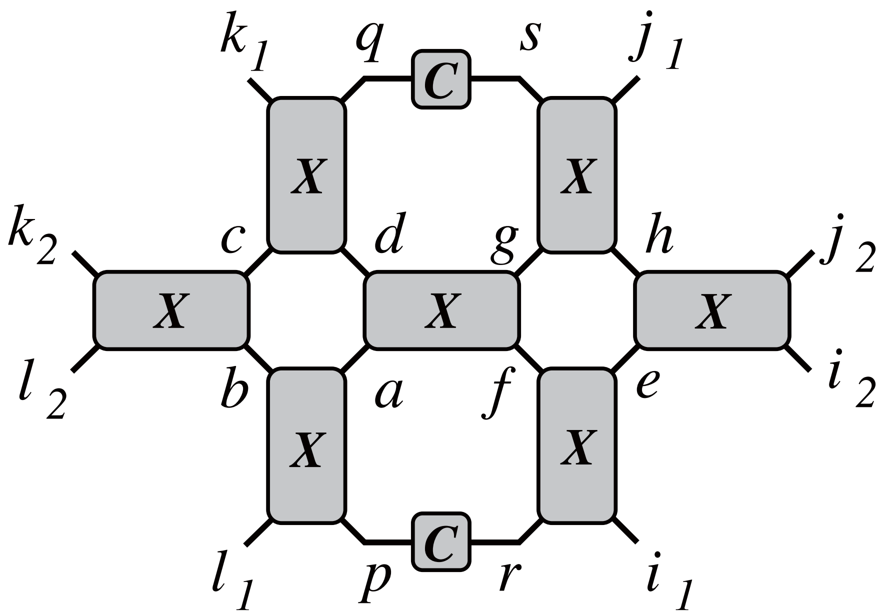
$$C^{(n+1)}_{ij} = C^{(n+1)}_{(i_1 i_2)(j_1 j_2)}$$

$$= \sum_{abcdef} C^{(n)}_{aj_2} X^{(n)}_{abcj_1} C^{(n)}_{fc} C^{(n)}_{db} X^{(n)}_{dei_1 f} C^{(n)}_{i_2 e}$$

New X

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)

X^{new} =



$$X^{(n+1)}_{ijkl} = X^{(n+1)}_{(i_1 i_2)(j_1 j_2)(k_1 k_2)(l_1 l_2)}$$
$$= \sum_{\substack{abcdef\\ghprqs}} X^{(n)}_{abl_1 p} X^{(n)}_{bck_2 l_2} X^{(n)}_{cdqk_1} X^{(n)}_{fgda}$$
$$X^{(n)}_{efri_1} X^{(n)}_{ghj_1 s} X^{(n)}_{i_2 j_2 he} C^{(n)}_{rp} C^{(n)}_{sq}$$

SVD of X

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)

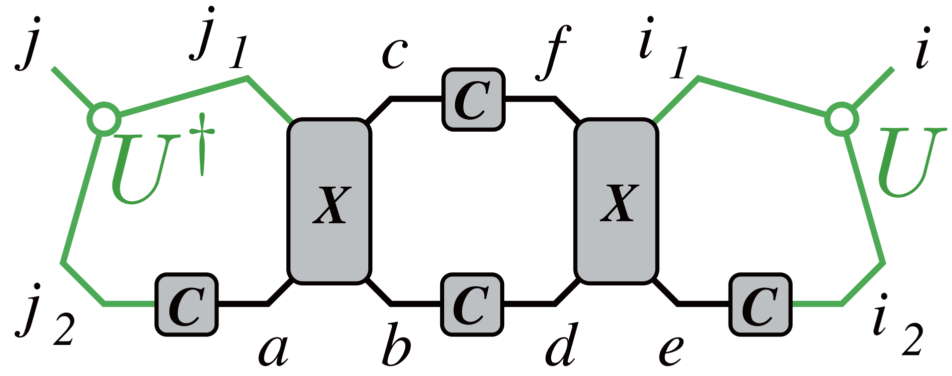
$$X^{(n+1)}_{(i_1 i_2)(j_1 j_2 k_1 k_2 l_1 l_2)} = \sum_{\xi} U_{(i_1 i_2)\,\xi}\, \omega_\xi\, V_{(j_1 j_2 k_1 k_2 l_1 l_2)\,\xi}$$

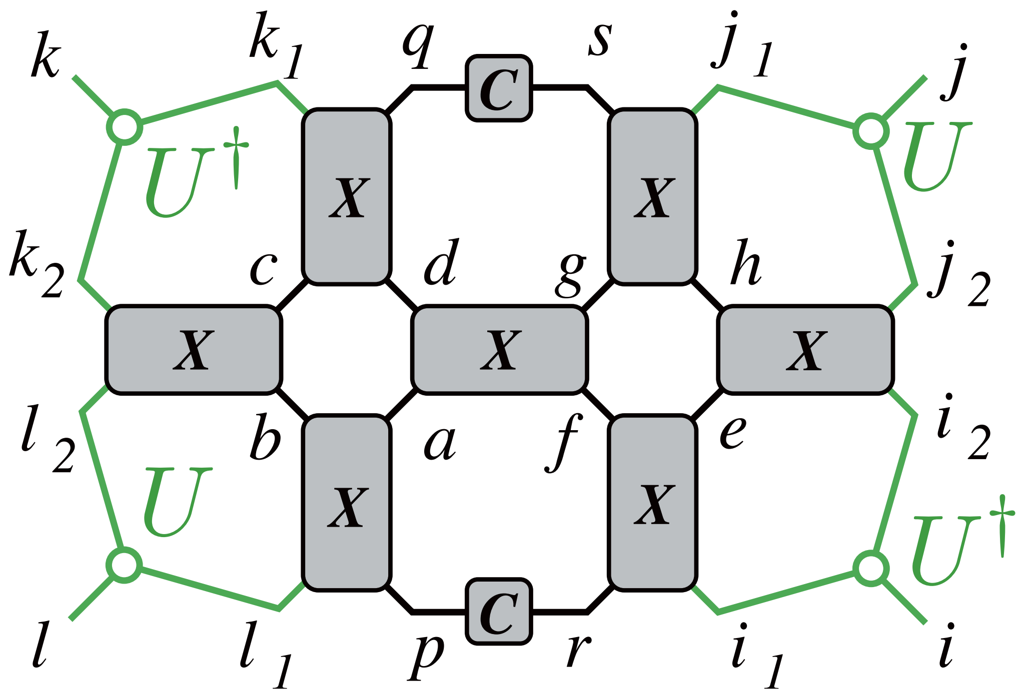
$$\simeq \sum_{i}^{D} U_{(i_1 i_2)\,i}\, \omega_i\, V_{(j_1 j_2 k_1 k_2 l_1 l_2)\,i}$$

U is used to truncate the bond dimension of X and C.

Renormalization of C and X

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)

$C^{new} =$ 

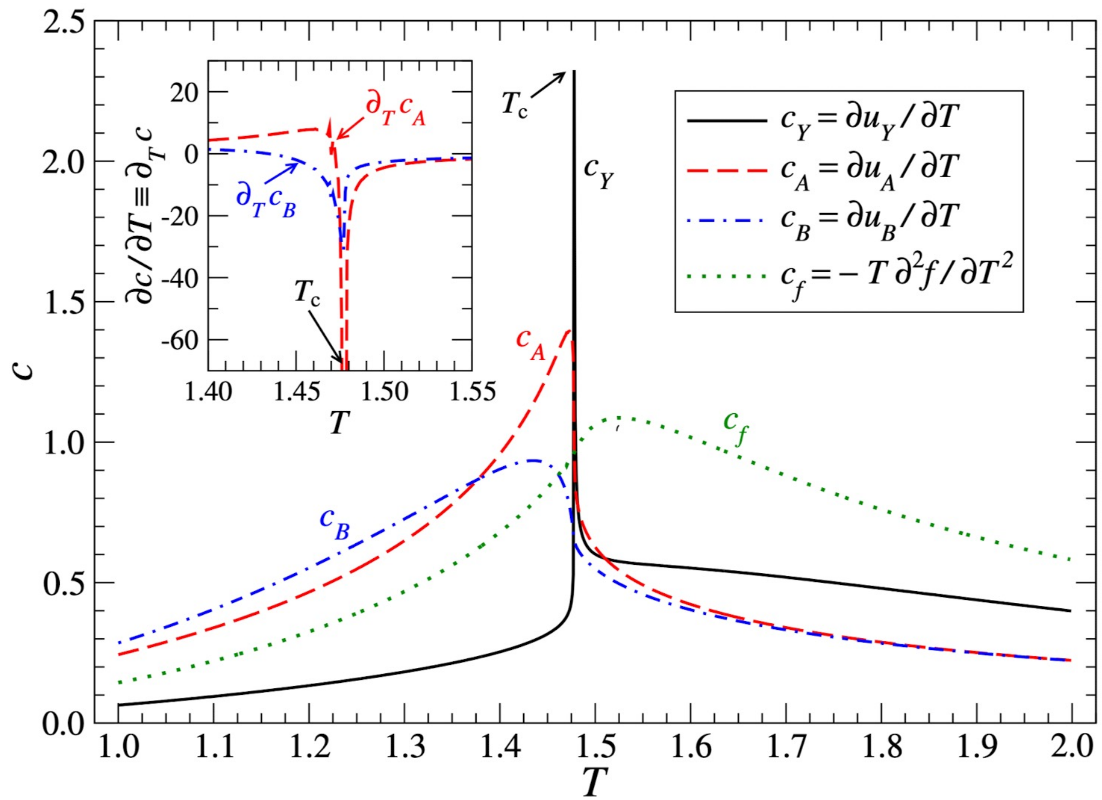
$X^{new} =$ 

3. Numerical results

Previous results

Jozef Genzor, Andrej Gendiar, Tomotoshi Nishino (2019)

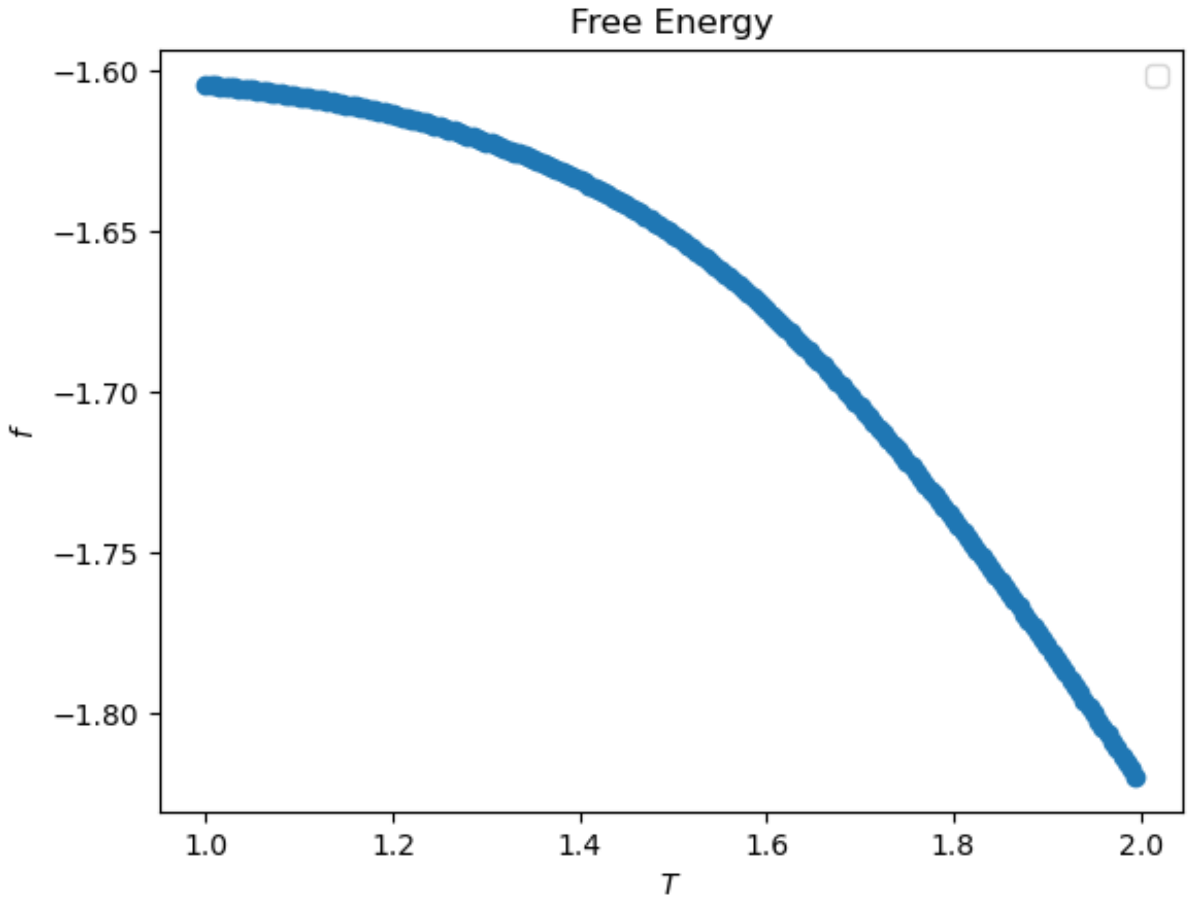
arxiv : 1904.10645



$n = 35$

Tc ≈1.47829

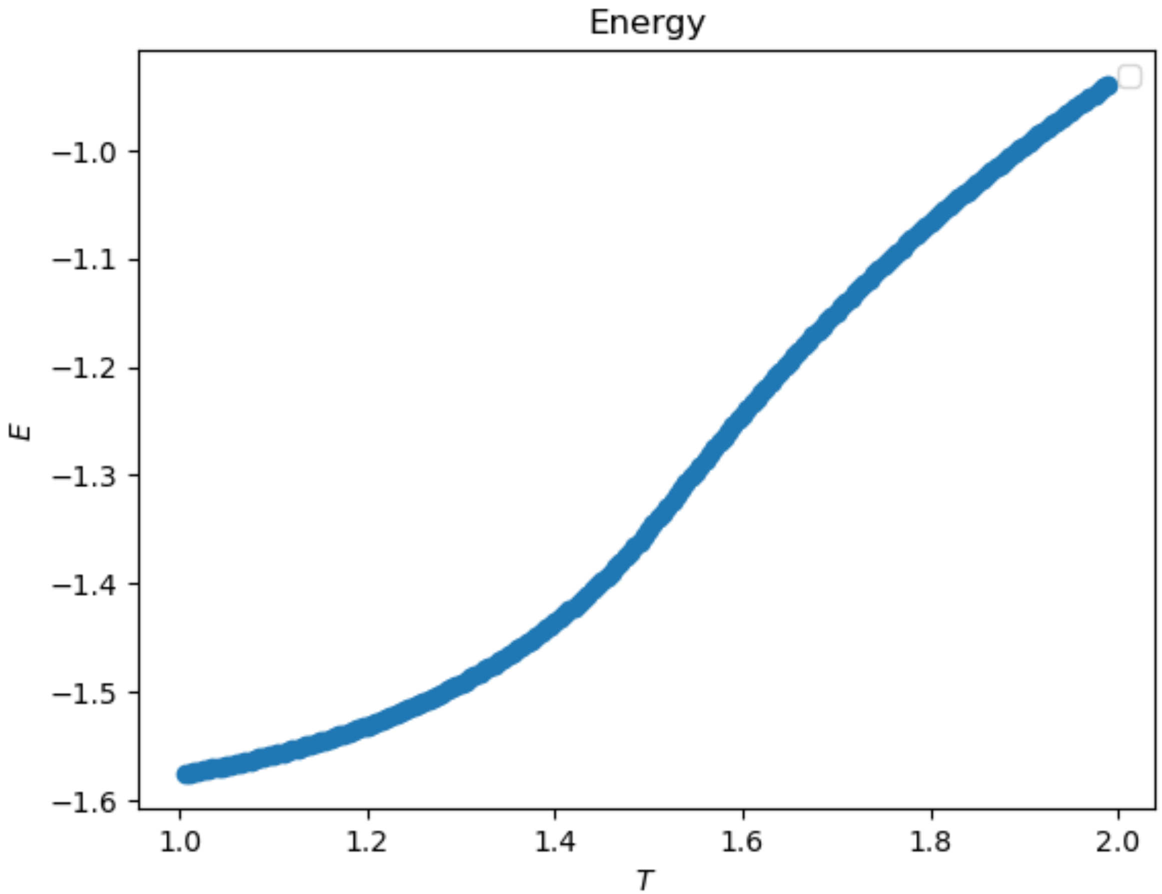
Free energy



$n = 35$

$D = 7$

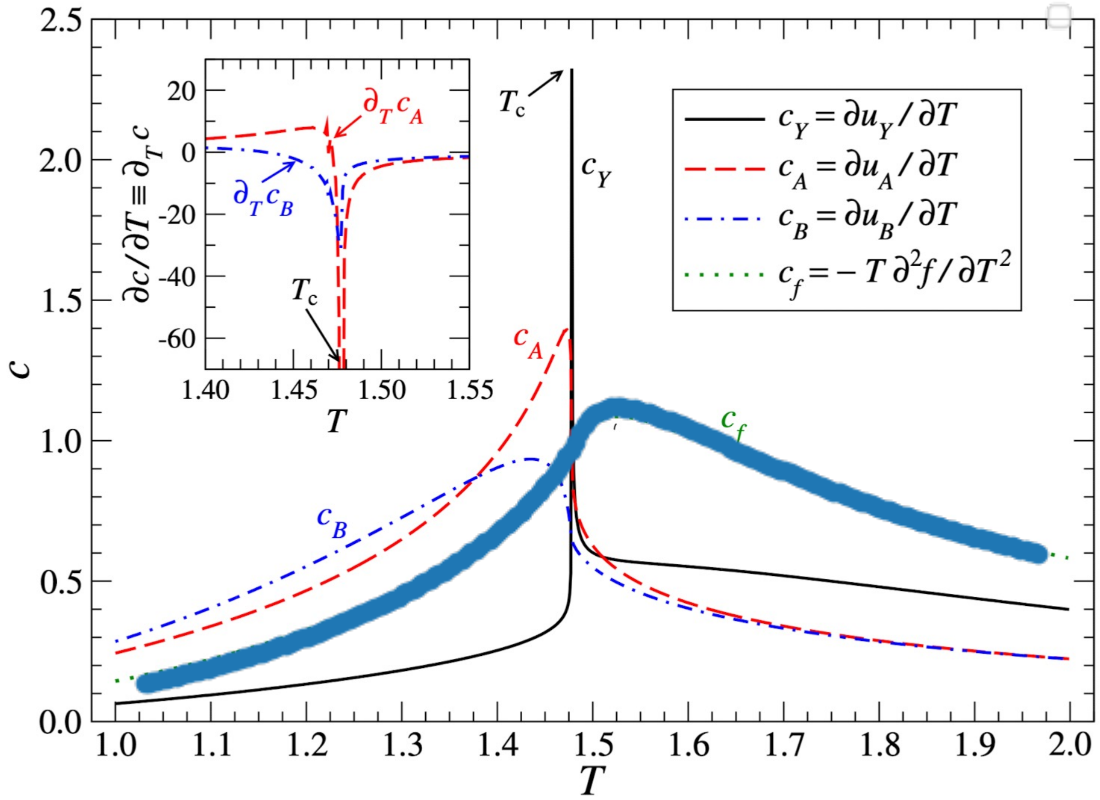
Internal Energy



$n = 35$

$D = 7$

Specific heat



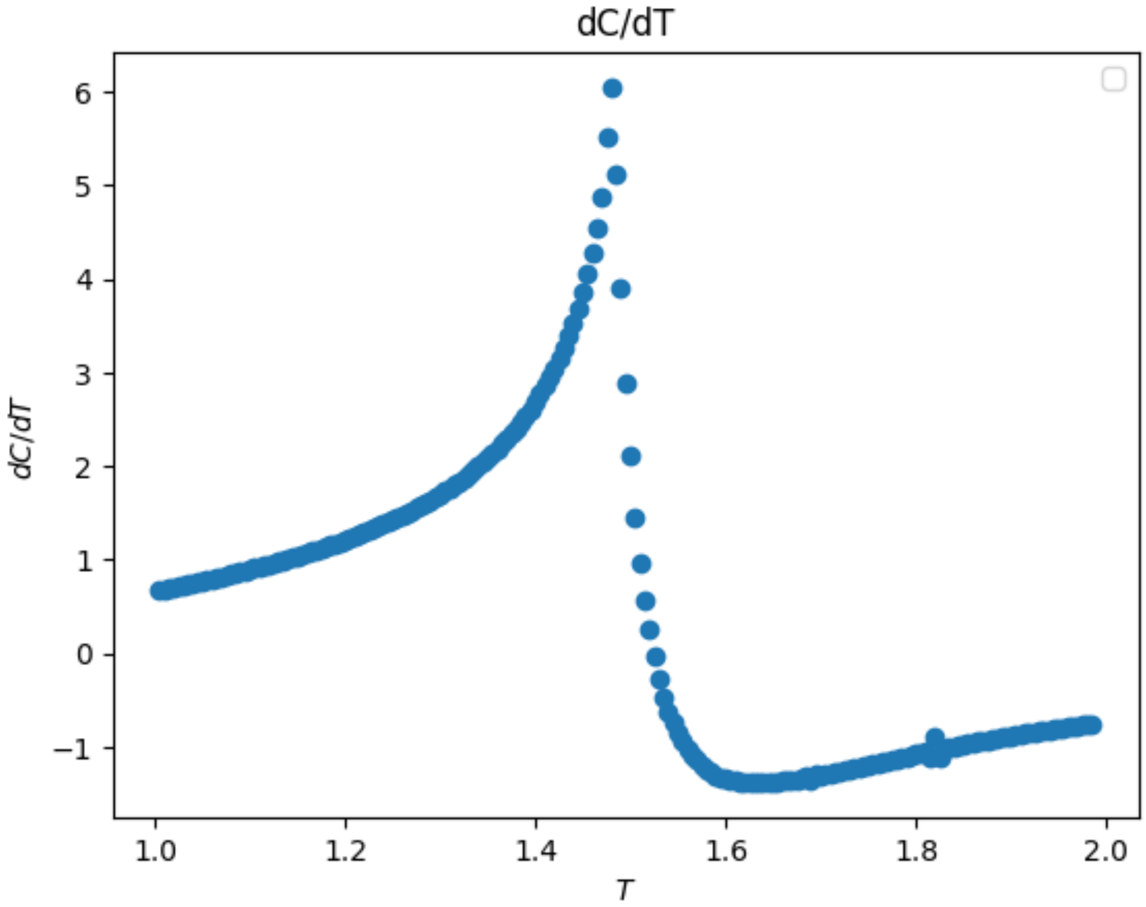
$n = 35$

$D = 7$

Successfully reproduced the results

dC/dT

Third-order phase transition?



$n = 35$

$D = 7$

Tc=1.48 is consistent with **Tc ≈1.47829** that was obtained in Genzor-Gendiar-Nishino(2019).

Summary

- We succeeded in reproducing the results of Genzor-Gendiar-Nishino (2019).
- We are in the process of improving the code to be able to calculate even larger bond dimensions.
- Other fractals
- Universality class, field theory on fractals?