Numerical Analysis of Entanglement Entropy for $1+1$ dimensional real scalar ϕ^4 theory using HOTRG

○ Takahiro Hayazaki[A] Daisuke Kadoh[B] Shinji Takeda[A] Gota Tanaka[B]

Kanazawa univ.[A], Meijigakuin univ.[B]

Tensor Network (2024) @ Kanazawa

November 22, 2024

Introduction

Entanglement Entropy (EE): Degree of entanglement
$\cdots$ Important value for critical phenomena

1. **EE is diverged on the quantum ciritical point**
2. **Extract central charge from EE**
   - **Introduced with CFT $\cdots$ Field Theory on cirical point**
   - **There are cases to obtain critical exponent in 2-dim.**

Purpose

Analysis using Tensor Renormalized Group

✓ **Numerical analysis of EE in the 2-dim. Ising model**

↓

Apply to 2-dim. scalar field theory

↓

Verify its usefulness in field theory

$1+1$-dim. real scalar field ϕ^4 Theory

Lattice Action

$$S_{\text{latt.}} = \sum_{n\in\Gamma}\left\{\frac{1}{2}\sum_{\rho=x,\tau}\left(\phi_{n+\hat{\rho}}-\phi_n\right)^2+\frac{\mu_0^2}{2}\phi_n^2+\frac{\lambda}{4}\phi_n^4\right\} \tag{1}$$

$\Gamma = \{n=(n_x,n_\tau)|n_\rho = 1,2,\cdots L_\rho\ (\rho = x,\tau)\}$,

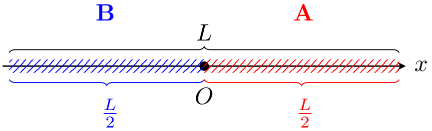
$\phi_{n+\hat{\rho}L_\rho} = \phi_n$: P.B.C.

μ_0: bare mass,

$\lambda > 0$: coupling const.

Definition of EE

- ✓ Separate the total system to 2 sub-systems for A & B
  - ▶ Hilbert Space: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$
  - ▶ Total system size: L
  - ▶ Sub-system size: $L/2$
- ✓ Total density matrix: ρ_{tot}
- ✓ Reduced density matrix for A system: $\rho_A \equiv \mathrm{Tr}_{\mathcal{H}_B} \rho_{tot}$



EE for A System

$$S_A \equiv -\mathrm{Tr}_{\mathcal{H}_A} [\rho_A \log \rho_A] \quad (2)$$

How to calculate EE

Density matrix calculation procedure [Luo and Kuramashi, 2023]

1. Express ρ_A as a path integral
2. Discretize the path integral and represent it as tensor network
   - Gauss-Hermite quadrature (K: # of sample points)
3. Coarse-graining using tensor renormalization group
   - HOTRG [Xie et al., 2012] (D_{cut}: Bond dimensions)
4. Calculate ρ_A using coarse-grained tensors

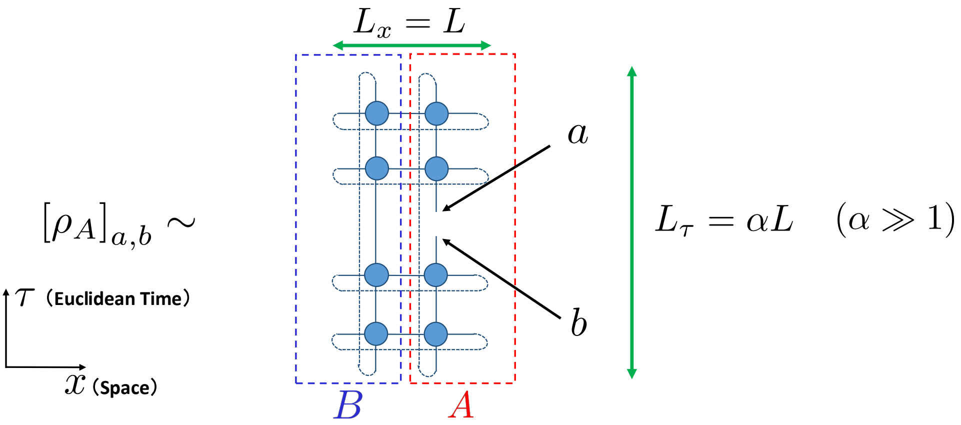
Parameters: $(\lambda, \mu_0^2), (K, L, D_{\mathrm{cut}})$

In the following, we will fix $K = 256$.

How to calculate EE

Tensor network representation of ρ_A at zero temperature [Yang et al., 2016]

$$[\rho_A]_{a,b} \sim$$

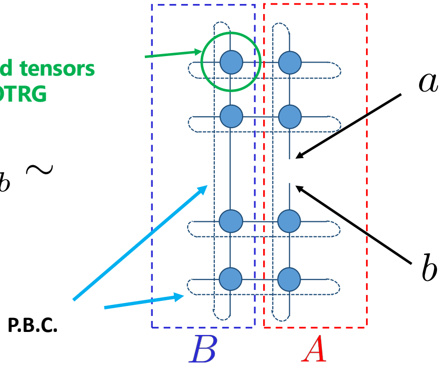


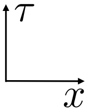
How to calculate EE

Tensor network representation of ρ_A at zero temperature [Yang et al., 2016]

Coarse-grained tensors using HOTRG

$$[\rho_A]_{a,b} \sim$$





Eigenvalue Decomposition

$$= \left[U \lambda U^\dagger\right]_{a,b}$$

$$S_A \sim -\sum_i \lambda_i \log \lambda_i$$

$$i = 1, 2, \cdots, D_{\mathrm{cut}}$$

Results

$$\lambda = 0.1, K = 256, \alpha = L_\tau / L_x = 16$$

Analysis procedure

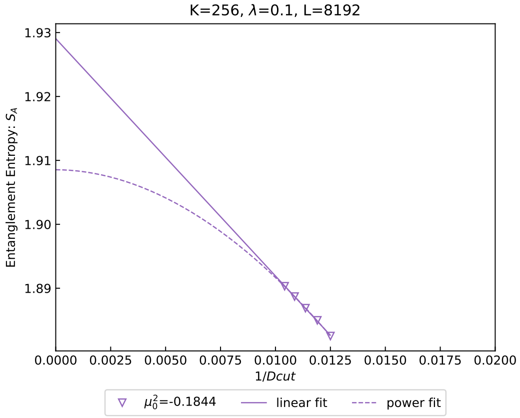
1. $D_{\text{cut}} \to \infty$ extrapolation of $S_A(L, \mu_0^2)$
2. Estimation of $\mu_{0,c}^2$ by S_A-μ_0^2 fitting
3. $L \to \infty$ extrapolation of renormalized μ_c^2
4. Calculating λ / μ_c^2

$D_{\text{cut}} \to \infty$ Extrapolation of EE

Fitting Function

$$f(D_{\text{cut}}^{-1}) = \begin{cases} a_0 D_{\text{cut}}^{-1} + a_1 \\ a_0' D_{\text{cut}}^{-a_2'} + a_1' \end{cases} \quad (3)$$

- ✓ D_{cut} dependence of EE
  - ▶ Monotonically increasing for $D_{\text{cut}} \to \infty$
- ✓ Error: $\frac{1}{2}\left|a_1 - a_1'\right|$
- ✓ Median: $\frac{1}{2}\left(a_1 + a_1'\right)$

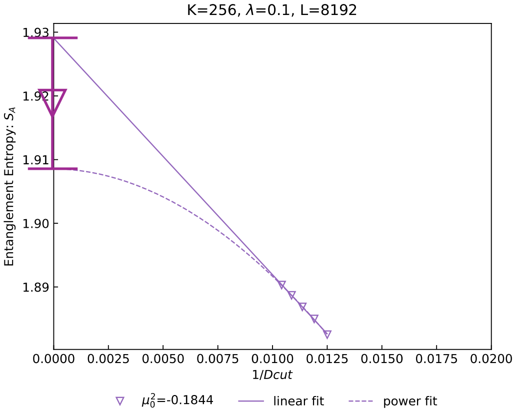


$D_{\mathrm{cut}} \to \infty$ Extrapolation of EE

Fitting Function

$$f(D_{\mathrm{cut}}^{-1}) = \begin{cases} a_0 D_{\mathrm{cut}}^{-1} + a_1 \\ a_0' D_{\mathrm{cut}}^{-a_2'} + a_1' \end{cases} \quad (3)$$

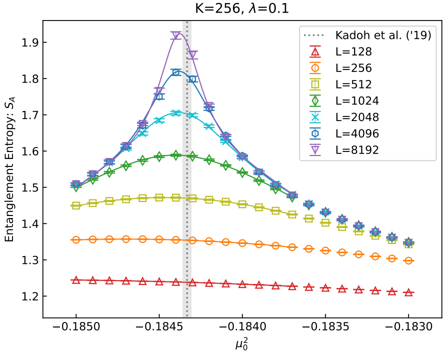
- ✓ D_{cut} dependence of EE
  - ▶ Monotonically increasing for $D_{\mathrm{cut}} \to \infty$
- ✓ Error: $\frac{1}{2}\left|a_1 - a_1'\right|$
- ✓ Median: $\frac{1}{2}\left(a_1 + a_1'\right)$



Fitting for S_A-μ_0^2

Fitting Function

$$f(\mu_0^2) = \frac{q_0 + q_1(\mu_0^2 - \mu_{0,c}^2)^2 + q_2(\mu_0^2 - \mu_{0,c}^2)^4}{1 + p_0(\mu_0^2 - \mu_{0,c}^2)^2} \quad (4)$$



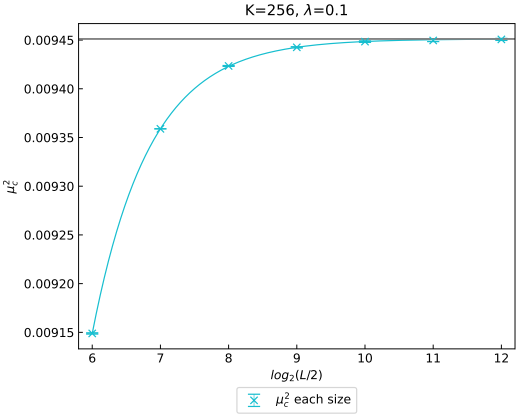
- ✓ Fitting parameters: $p_0, q_0, q_1, q_2, \mu_{0,c}^2$
- ✓ $\mu_{0,c}^2$: The value of μ_0^2 at which EE peak
- ✓ q_0 : Peak EE value

$L \to \infty$ extrapolation of renormalized mass: μ_c^2

Fitting Function

$$\mu_c^2(L) = d_0 \times \left(\frac{L}{2}\right)^{-d_2} + d_1 \qquad (5)$$

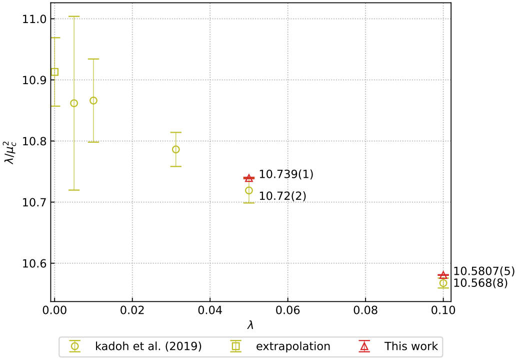
- ✓ $L/2 = 2^6, 2^7, \cdots, 2^{12}$
- ✓ This error is due to $D_{\text{cut}} \to \infty$ extrapolation.
  - ▶ Obtain extrapolated values with error



Result for λ/μ_c^2

- ✓ Results for $\lambda = 0.1, 0.05$ (RD)
- ✓ [Kadoh et al., 2019] results (YL)
- ✓ Update λ/μ_c^2 for $\lambda = 0.1, 0.05$

λ	$[\lambda/\mu_c^2]_{\mathrm{YL}}$	$[\lambda/\mu_c^2]_{\mathrm{RD}}$
0.1	$10.568(8)$	$10.5807(5)$
0.05	$10.72(2)$	$10.739(1)$

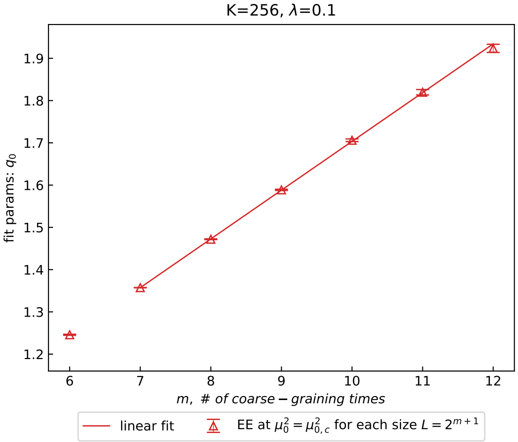


Result for central charge

Fitting Function

$$f(m) = b_0 \times m + b_1$$
$$c = \frac{3}{\ln 2} \times b_0 \quad (6)$$

- ✓ $S_A = \frac{c}{3} \ln L + k$
- ✓ c: central charge
- ✓ k: constant independent of L
- ✓ x-axis: m, # of coarse-graining times (size $L = 2^{m+1}$)
- ✓ y-axis: EE at $\mu_0^2 = \mu_{0,c}^2$

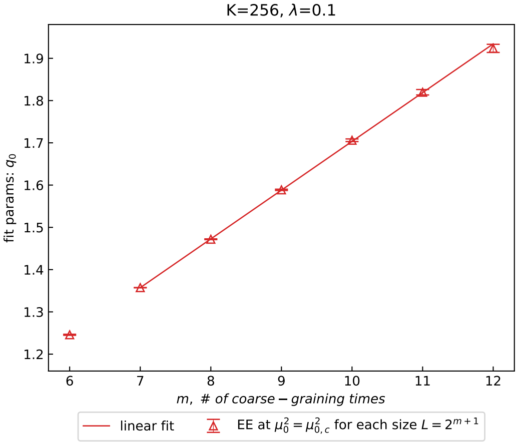


Result for central charge

Fitting Function

$$f(m) = b_0 \times m + b_1$$
$$c = \frac{3}{\ln 2} \times b_0 \quad (6)$$

- ✓ $S_A = \frac{c}{3} \ln L + k$
- ✓ $c_{\text{calc.}} = 0.499(2)$
- ✓ $c_{\text{exact}} = 0.5$



Summary

- $D\text{cut}$ dependence of EE
  - Monotonically increasing with respect to D_{cut}
- μ_0^2 dependence of EE
  - As the size increases, a tendency for divergence is observed near the critical point.
- Update λ/μ_c^2 for $\lambda = 0.1, 0.05$

Future Work

- analyze for smaller λ, and extrapolate $\lambda \to 0$
- Compare the accuracy by calculating $\mu_{0,c}^2$ using other calculation methods
  - Method using X
  - Impurity Tensor Method
- Two-dimensional fermion model
- Two-dimensional gauge theory
- Extend to 3D and 4D models

Reference

[Kadoh et al., 2019] Kadoh, D., Kuramashi, Y., Nakamura, Y., Sakai, R., Takeda, S., and Yoshimura, Y. (2019).
Tensor network analysis of critical coupling in two dimensional ϕ^4 theory.
Journal of High Energy Physics, 2019(5).

[Luo and Kuramashi, 2023] Luo, X. and Kuramashi, Y. (2023).
Entanglement and rényi entropies of (1+1)-dimensional o(3) nonlinear sigma model with tensor renormalization group.

[Xie et al., 2012] Xie, Z. Y., Chen, J., Qin, M. P., Zhu, J. W., Yang, L. P., and Xiang, T. (2012).
Coarse-graining renormalization by higher-order singular value decomposition.
Physical Review B, 86(4).

[Yang et al., 2016] Yang, L.-P., Liu, Y., Zou, H., Xie, Z. Y., and Meurice, Y. (2016).
Fine structure of the entanglement entropy in the o(2) model.
Phys. Rev. E, 93:012138.

Appendix A: Path integral representation of ρ_A

$$[\rho_A]_{\phi_-^A,\phi_+^A} = \frac{1}{Z}\left[\int \prod_{\substack{-\infty<\tau<-0,\\ +0<\tau<\infty}} \prod_{x\in A} d\phi(\tau,x)\; e^{-S_E[\phi]}\right] \delta_A[\phi_{-0}-\phi_-^A]\cdot\delta_A[\phi_{+0}-\phi_+^A] \tag{A.1}$$

where

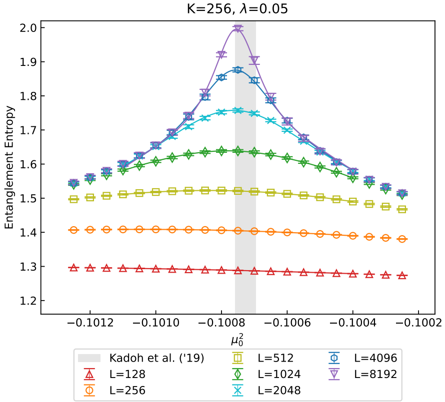
$$\delta_A\left[\phi_0-\phi\right] = \prod_{x\in A}\delta(\phi_0(x)-\phi(x)), \quad \phi_{\pm 0}\equiv\phi(\tau=\pm 0),$$

$$Z = \int\left[\prod_{-\infty<\tau<\infty}\prod_{x_1} d\phi(\tau,x)\right] e^{-S_E[\phi]}$$

Appendix B: Fitting for $S_A - \mu_0^2$ $(\lambda = 0.05)$

Fitting Function

$$f(\mu_0^2) = \frac{q_0 + q_1(\mu_0^2 - \mu_{0,c}^2)^2 + q_2(\mu_0^2 - \mu_{0,c}^2)^4}{1 + p_0(\mu_0^2 - \mu_{0,c}^2)^2} \quad (4)$$



- ✓ Fitting parameters: $p_0, q_0, q_1, q_2, \mu_{0,c}^2$
- ✓ $\mu_{0,c}^2$: The value of μ_0^2 at which EE peaks
- ✓ q_0 : Peak EE value

Appendix C: Calculation of μ^2

1-loop correction

$$\mu^2 = \mu_0^2 + 3\lambda A(\mu^2)$$
$$A(\mu^2) = \frac{1}{L^2}\sum_{k_1=1}^{L}\sum_{k_2=1}^{L}\frac{1}{\mu^2 + 4\sin^2(\pi k_1/L) + 4\sin^2(\pi k_2/L)} \quad \text{(A.4)}$$

- ✓ Self Consistent Eq.
  - Calculation using False Position Method
- ✓ Stopping condition
  - $f(\mu^2) \equiv \mu^2 - \mu_0^2 - 3\lambda A(\mu^2)$
  - $\left|f(\mu^2)\right| < \epsilon = 10^{-10}$