DMRG calculation of the θ-dependent mass spectrum in the 2-flavor Schwinger model

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collaboration with
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JHEP09 (2024) 155 [2407.11391]

Tensor Network 2024, 16 November 2024 @Kanazawa

Background: mass spectrum of QCD

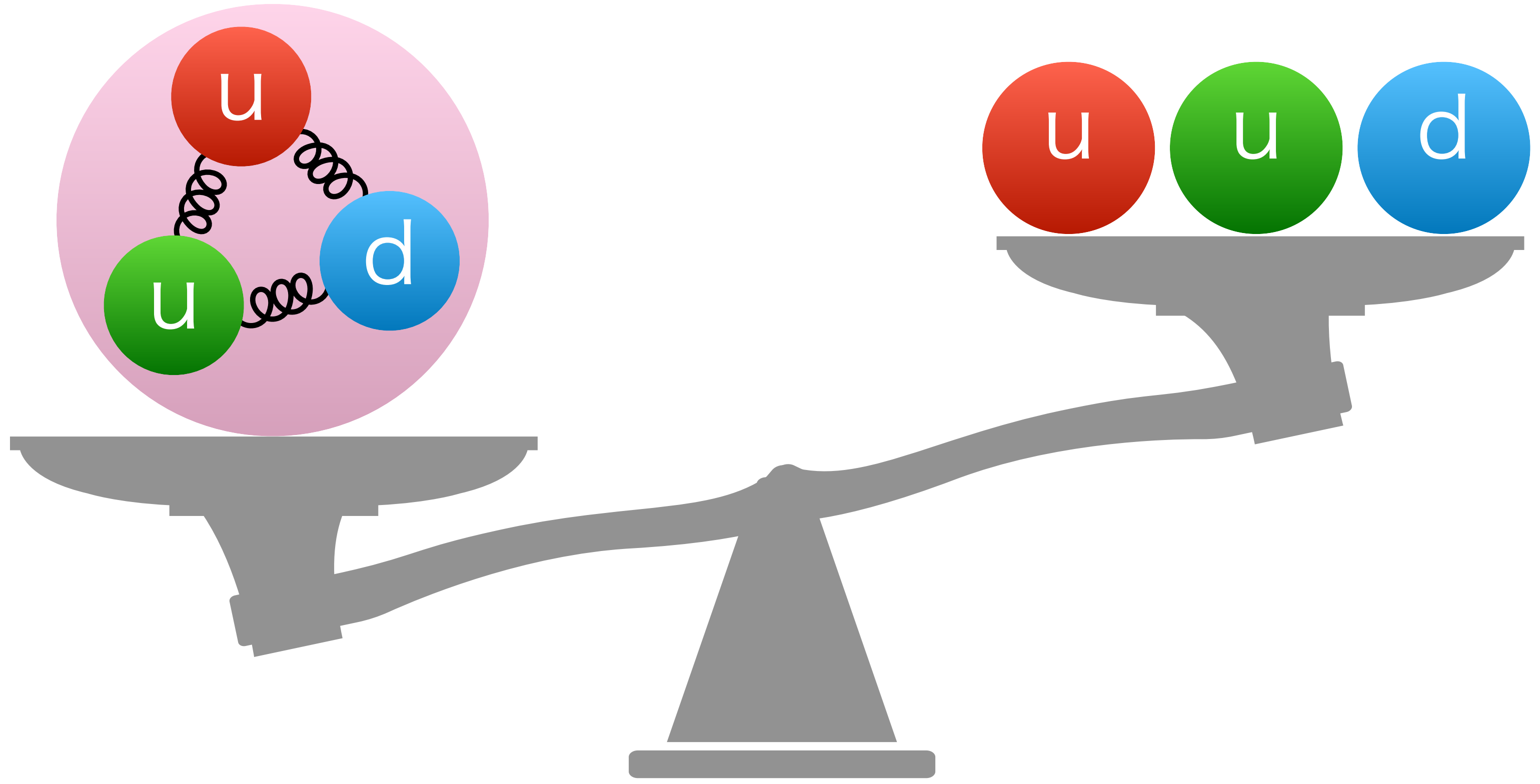
- quark confinement in Quantum ChromoDynamics (QCD)
  ··· low-energy d.o.f. are not quarks but composite particles (hadrons)

- **hadrons are much heavier than quarks**

  u/d quark: $m_u \sim 2$ MeV, $m_d \sim 5$ MeV

  π^+ meson (u, d): $140 \text{ MeV} \gg m_u + m_d$

  proton (u, u, d): $938 \text{ MeV} \gg 2m_u + m_d$



- nonperturbative calc. is essential to understand the properties of hadrons

motivation:
Numerically investigate low-energy spectra of gauge theories such as QCD

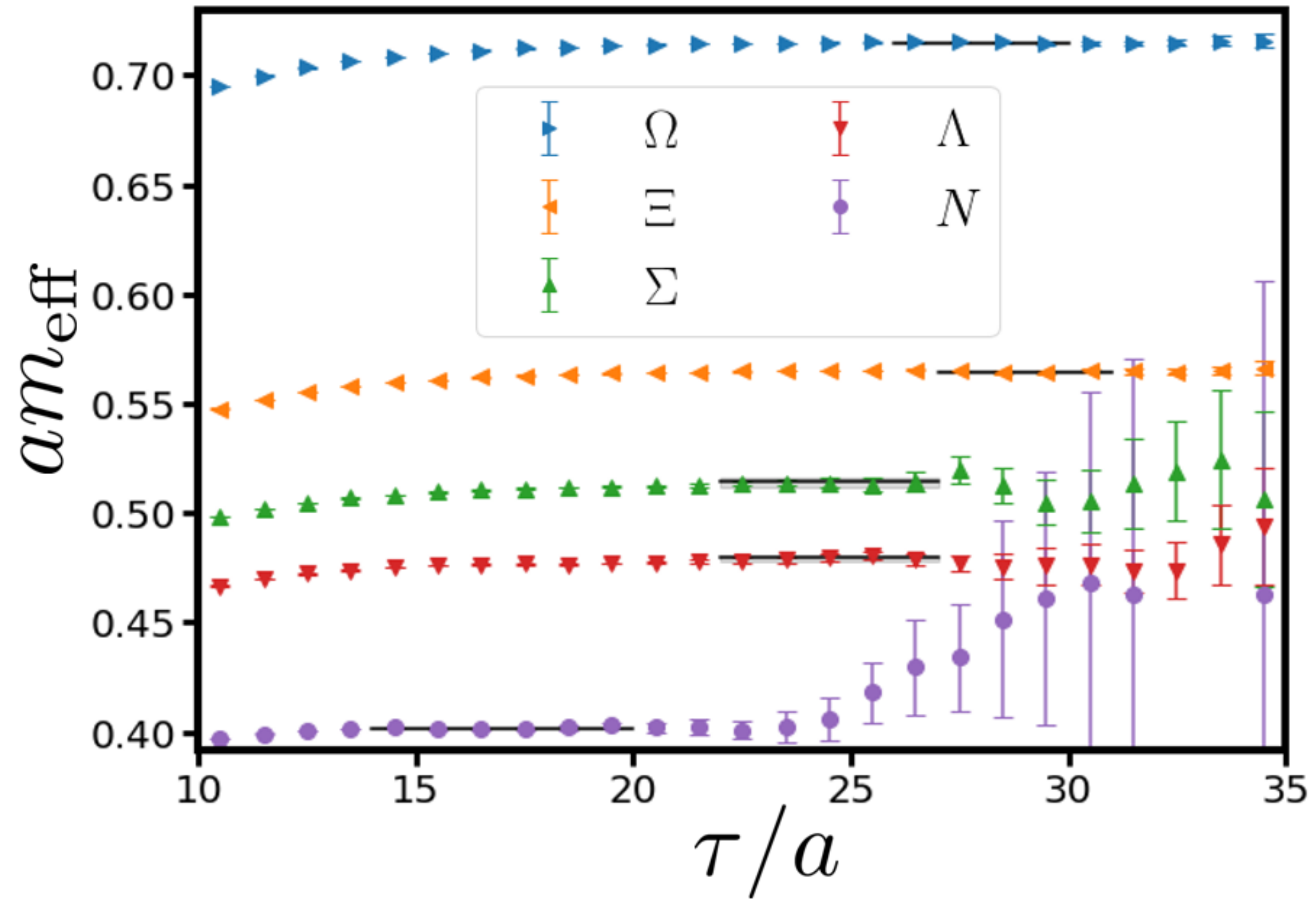
Mass spectrum by lattice QCD

- well-established method:
  Monte Carlo simulation of the lattice gauge theory (Lagrangian formalism)

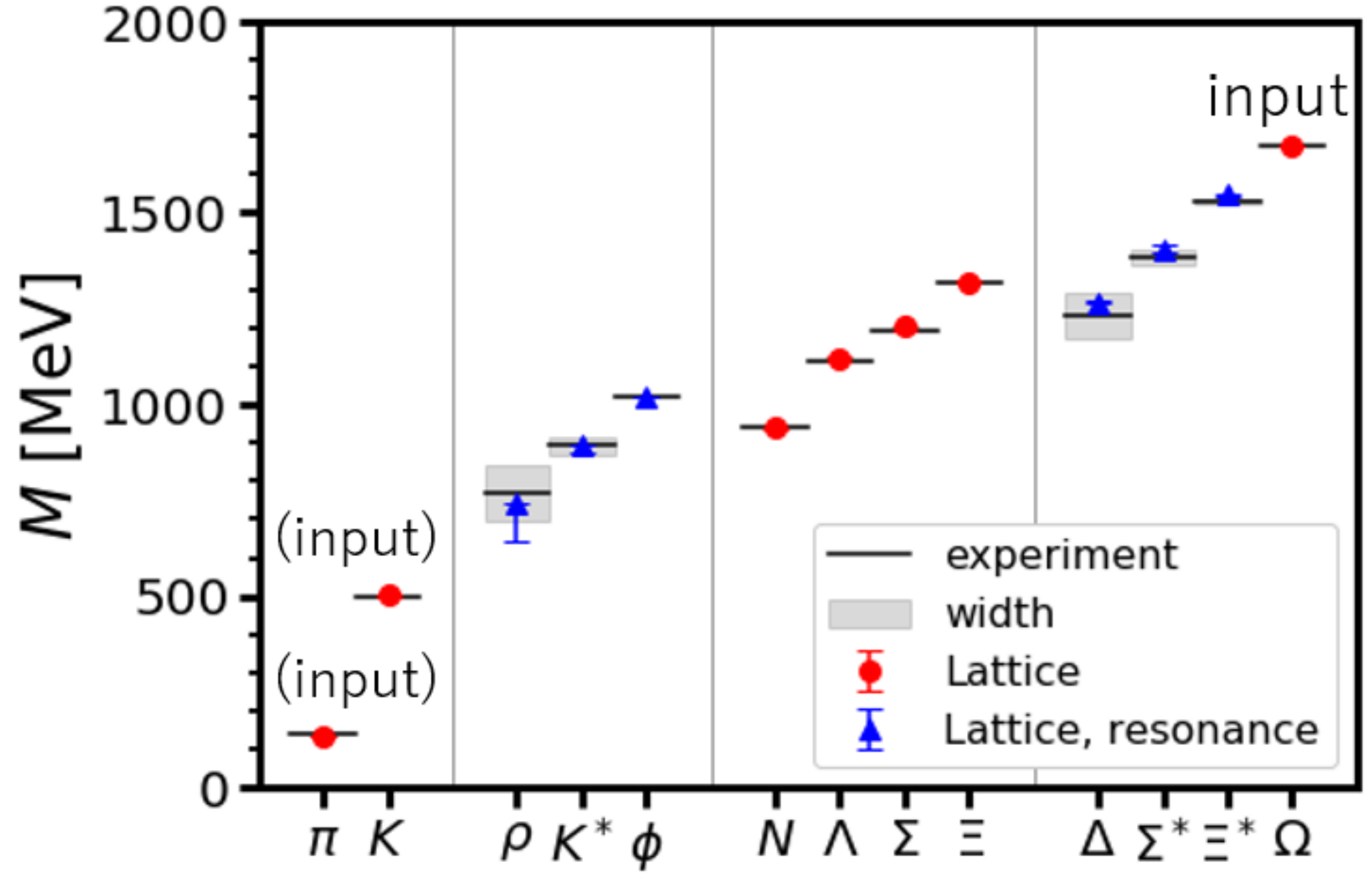
- obtain hadron masses from imaginary-time correlation functions

$$C(\tau) = \sum_x \langle \mathcal{O}(0,0)\, \mathcal{O}(x,\tau) \rangle \sim e^{-M\tau} \quad \text{—>} \quad \text{effective mass: } m_{\text{eff}}(\tau) = -\log \frac{C(\tau+1)}{C(\tau)} \simeq M$$

effective mass



hadron spectrum



[HAL QCD collab. (2024)]

Hamiltonian formalism

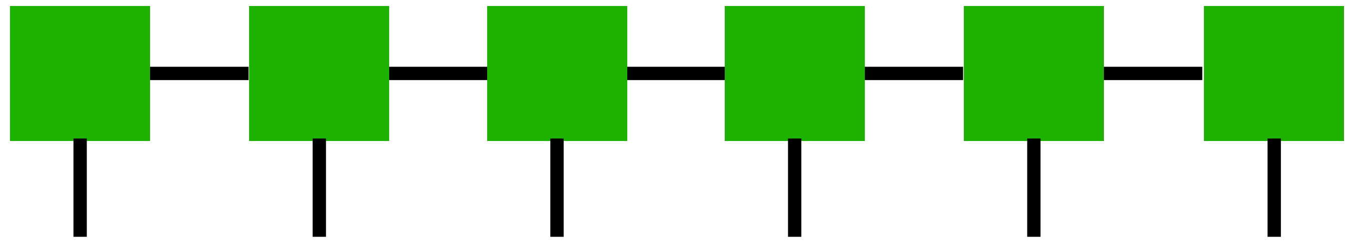
😣 Monte Carlo method cannot be applied to models with complex actions

—> sign problem (finite density QCD, topological term, real-time evolution, ⋯)

💡 Tensor network and quantum computing in Hamiltonian formalism
can be complementary approaches!

👍 free from the sign problem

👍 analyze excited states directly



aim of this work:
computing the hadron mass spectrum
in Hamiltonian formalism that is applicable
even when the sign problem arises

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Short summary

- demonstrate three distinct methods to compute the mass spectrum of the 2-flavor Schwinger model at $\theta = 0$ JHEP11 (2023) 231

  (1) correlation-function scheme

  (2) one-point-function scheme

  (3) dispersion-relation scheme

- improve and extend them to the case of $\theta \neq 0$ JHEP09 (2024) 155

  (1)+(2) improved one-point-function scheme

  (3) dispersion-relation scheme

- θ-dependent spectra by these schemes are consistent with each other and with calculation in the bosonized model

Outline

1. 2-flavor Schwinger model and calculation strategy
2. Improved one-point-function scheme
3. Dispersion-relation scheme
4. Summary

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Schwinger model with two fermions

<u>Schwinger model = Quantum ElectroDynamics in 1+1d</u>

- simplest nontrivial gauge theory sharing some features with QCD

$$\mathscr{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_f}\left[i\bar{\psi}_f\gamma^\mu\left(\partial_\mu + iA_\mu\right)\psi_f - m\bar{\psi}_f\psi_f\right]$$

sign problem if $\theta \neq 0$

- quantum numbers:

  isospin J, parity P, G-parity $G = Ce^{i\pi J_y}$

- P and G are broken at $\theta \neq 0$

  —> η becomes unstable

  due to $\eta \to \pi\,\pi$ decay and η - σ mixing

$N_f = 2$ —> three "mesons"

$\pi_a = -i\bar{\psi}\gamma^5\tau_a\psi$: $J^{PG} = 1^{-+}$

$\sigma = \bar{\psi}\psi$: $J^{PG} = 0^{++}$

$\eta = -i\bar{\psi}\gamma^5\psi$: $J^{PG} = 0^{--}$

Calculation strategy

[Kogut & Susskind (1975)]
[Dempsey et al. (2022)]

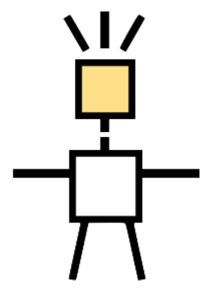
- setup: staggered fermion with open boundary

$$H=\frac{g^2a}{2}\sum_{n=0}^{N-2}\left(L_n+\frac{\theta}{2\pi}\right)^2+\sum_{f=1}^{N_f}\left[\frac{-i}{2a}\sum_{n=0}^{N-2}\left(\chi_{f,n}^{\dagger}U_n\chi_{f,n+1}-\chi_{f,n+1}^{\dagger}U_n^{\dagger}\chi_{f,n}\right)+m_{\mathrm{lat}}\sum_{n=0}^{N-1}(-1)^n\chi_{f,n}^{\dagger}\chi_{f,n}\right]$$

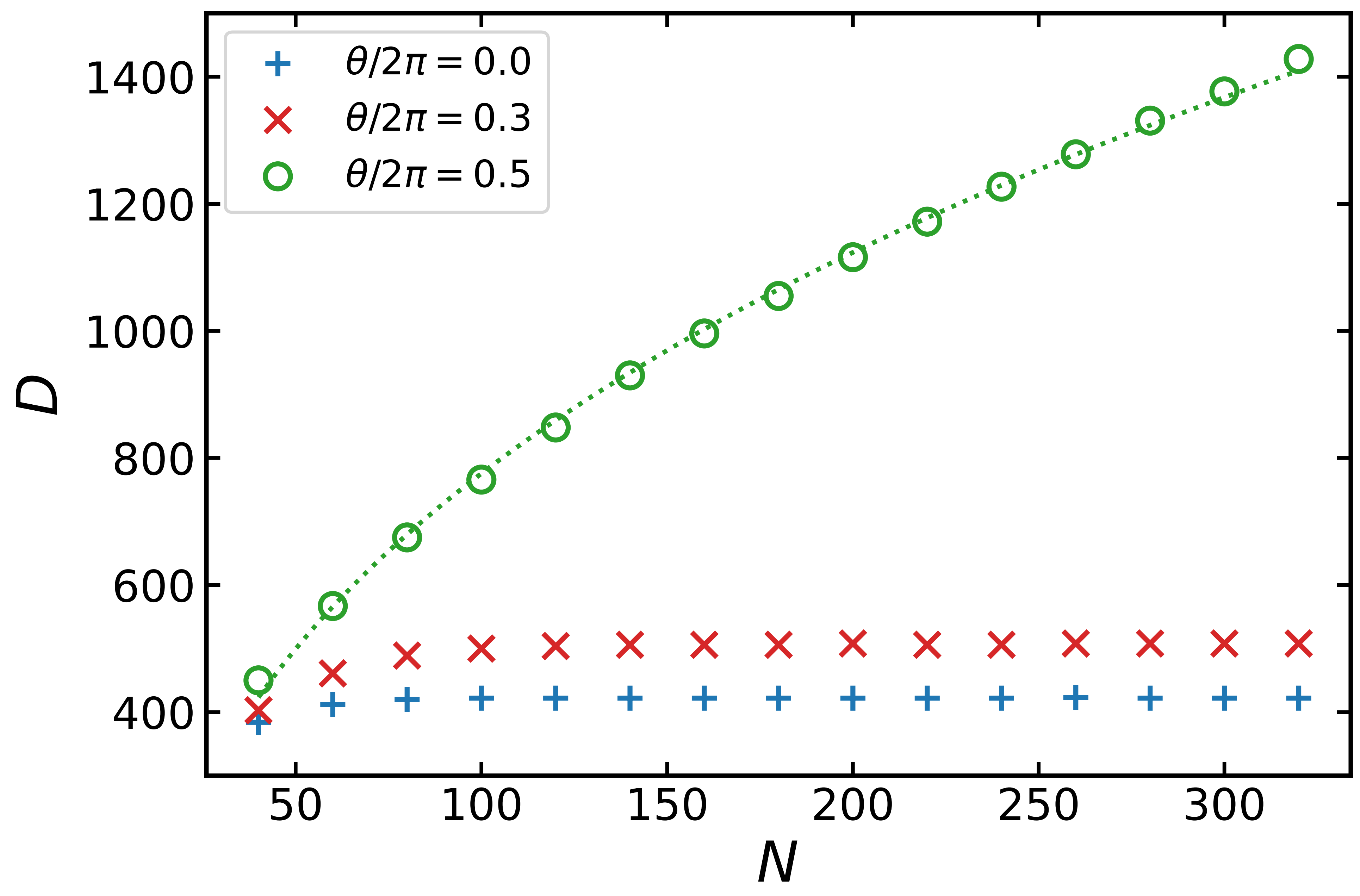
- rewrite to the spin Hamiltonian by Jordan-Wigner transformation after solving Gauss law and gauge fixing

- obtain the ground state as MPS by density-matrix renormalization group (DMRG)

C++ library of ITensor is used
[Fishman et al. (2022)]



bond dim. for fixed truncation error



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1. 2-flavor Schwinger model and calculation strategy
2. **Improved one-point-function scheme**
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Correlation function?

- correlation function with spatial integral in lattice QCD

  —> zero-momentum projection: $\sum_x \langle \mathcal{O}(0,0)\mathcal{O}(x,\tau)\rangle \sim e^{-M\tau}$

- equal-time correlator in Hamiltonian formalism

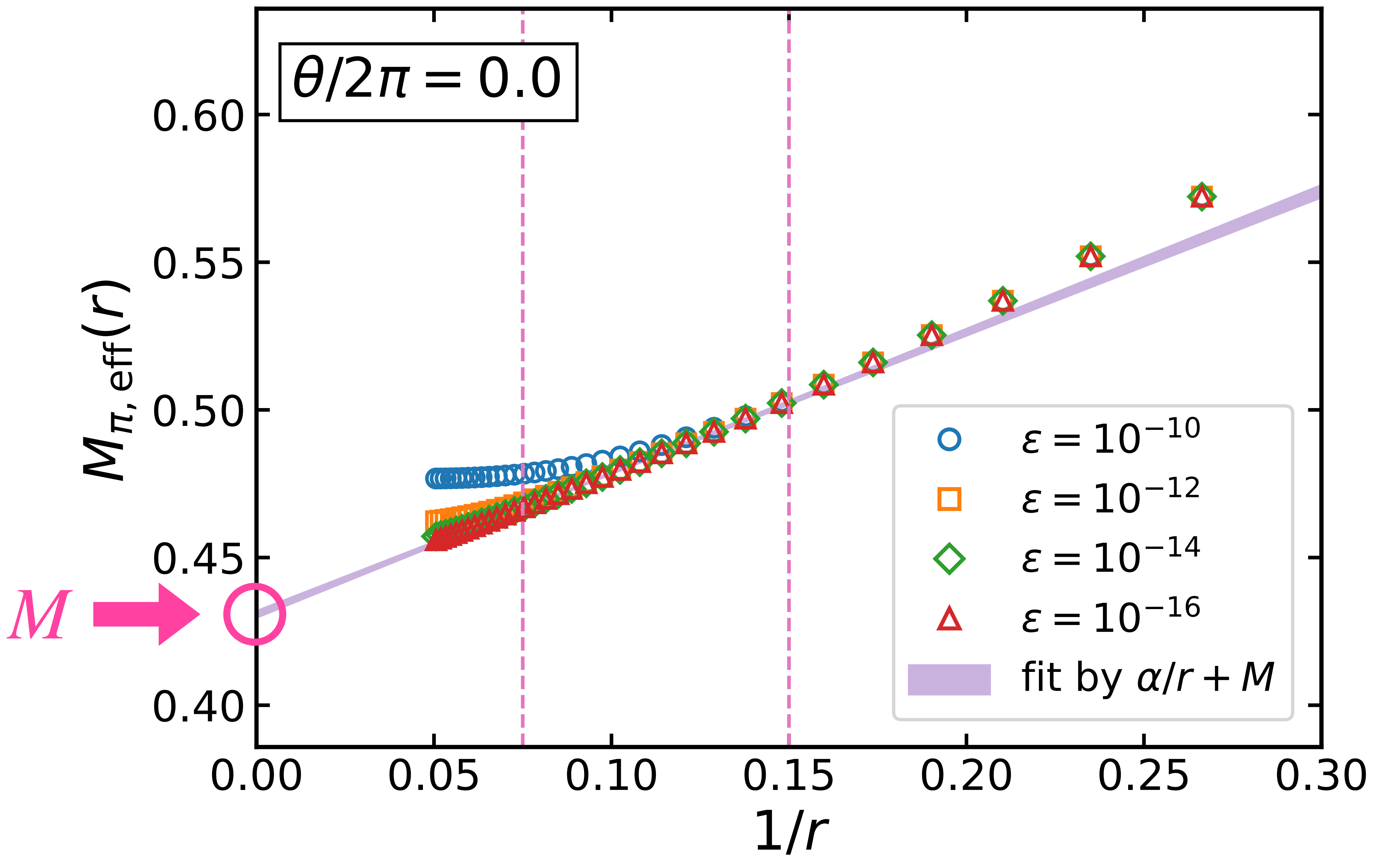
  —> $C(r) = \langle \mathcal{O}(0,0)\, \mathcal{O}(r,0)\rangle \sim \frac{1}{r^\alpha}e^{-Mr}$

  —> effective mass: $M_{\rm eff}(r) = -\frac{d \log C(r)}{dr} \sim \frac{\alpha}{r} + M$

- bond dim. must be large enough to see $1/r$ behavior

⚠️ significant truncation effect

effective mass of π meson



One-point-function scheme

<u>Regarding the boundary (defect) as the source of mesons, obtain the masses from the one-point functions</u>

- 1pt. function $\langle \mathcal{O}(x) \rangle_{\mathrm{obc}}$ measures the correlation with the boundary state $|\mathrm{bdry}\rangle$
- $|\mathrm{bdry}\rangle$ has translational invariance in time direction

  —> zero-momentum projection —> exponential decay

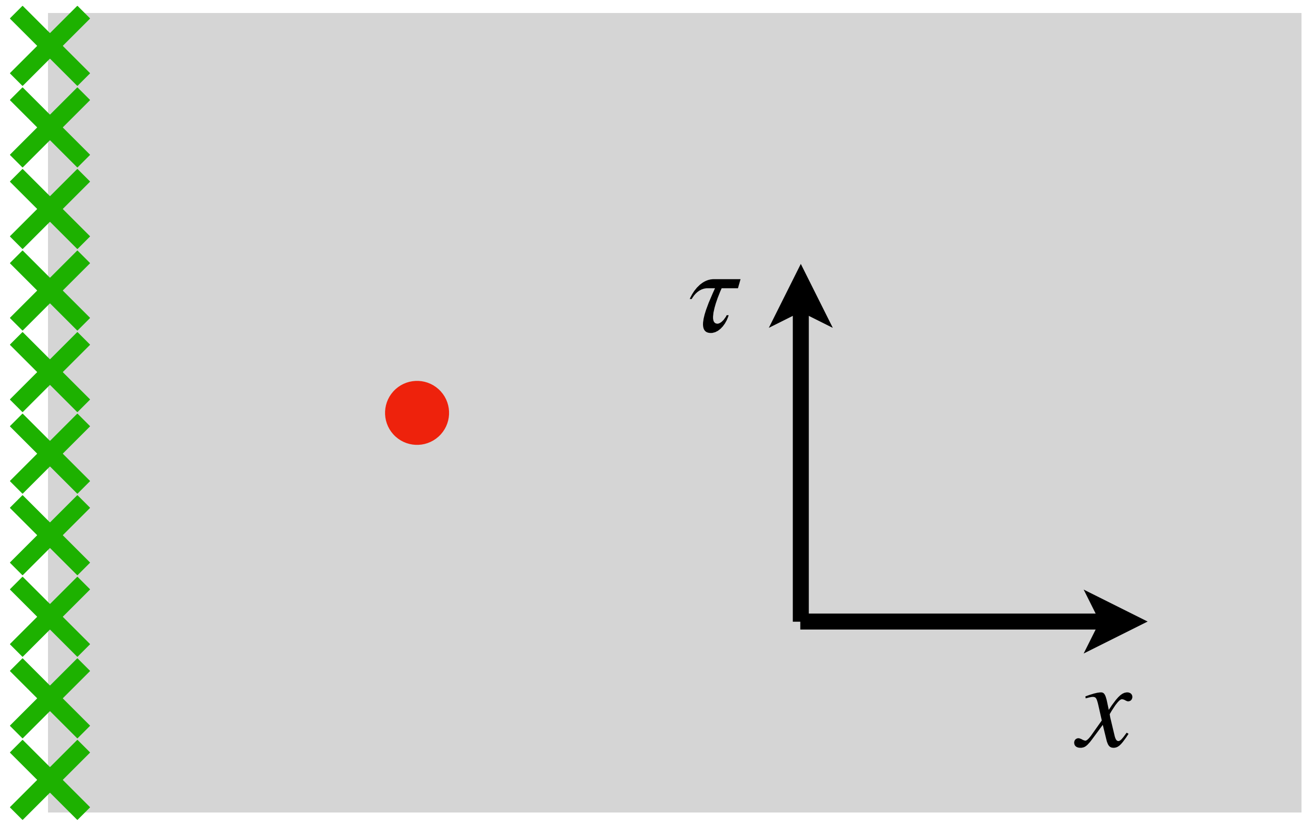
$$p_\tau |\mathrm{bdry}\rangle = 0$$

$$\langle \mathcal{O}(x) \rangle_{\mathrm{obc}} \sim \langle \mathrm{bdry} | e^{-Hx} \mathcal{O} | 0 \rangle_{\mathrm{bulk}} \sim e^{-Mx}$$

boundary state

$\mathcal{O}(x)$

Euclidean space



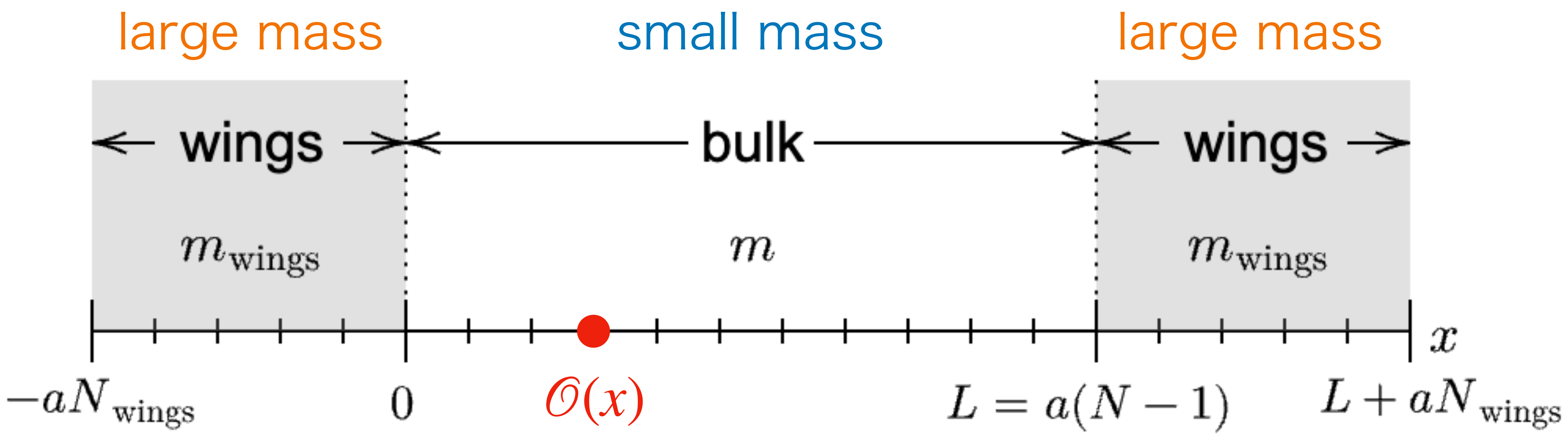
👍 truncation effect is much smaller

cf.) wall source method

Some technical improvement

- We attach “the wings” to the lattice to control the boundary condition flexibly

  e.g.) Dirichlet b.c. $\cdots$ $m_{\text{wings}} \gg m$



- The boundary must have the same quantum number as the target meson

Result of sigma and eta mesons

- For the singlet mesons, we set the Dirichlet b.c. $m_{\text{wings}} = m_0 \gg m$
- Assuming $\langle \mathcal{O}(x) \rangle \sim Ae^{-Mx} + Be^{-(M+\Delta M)x}$,

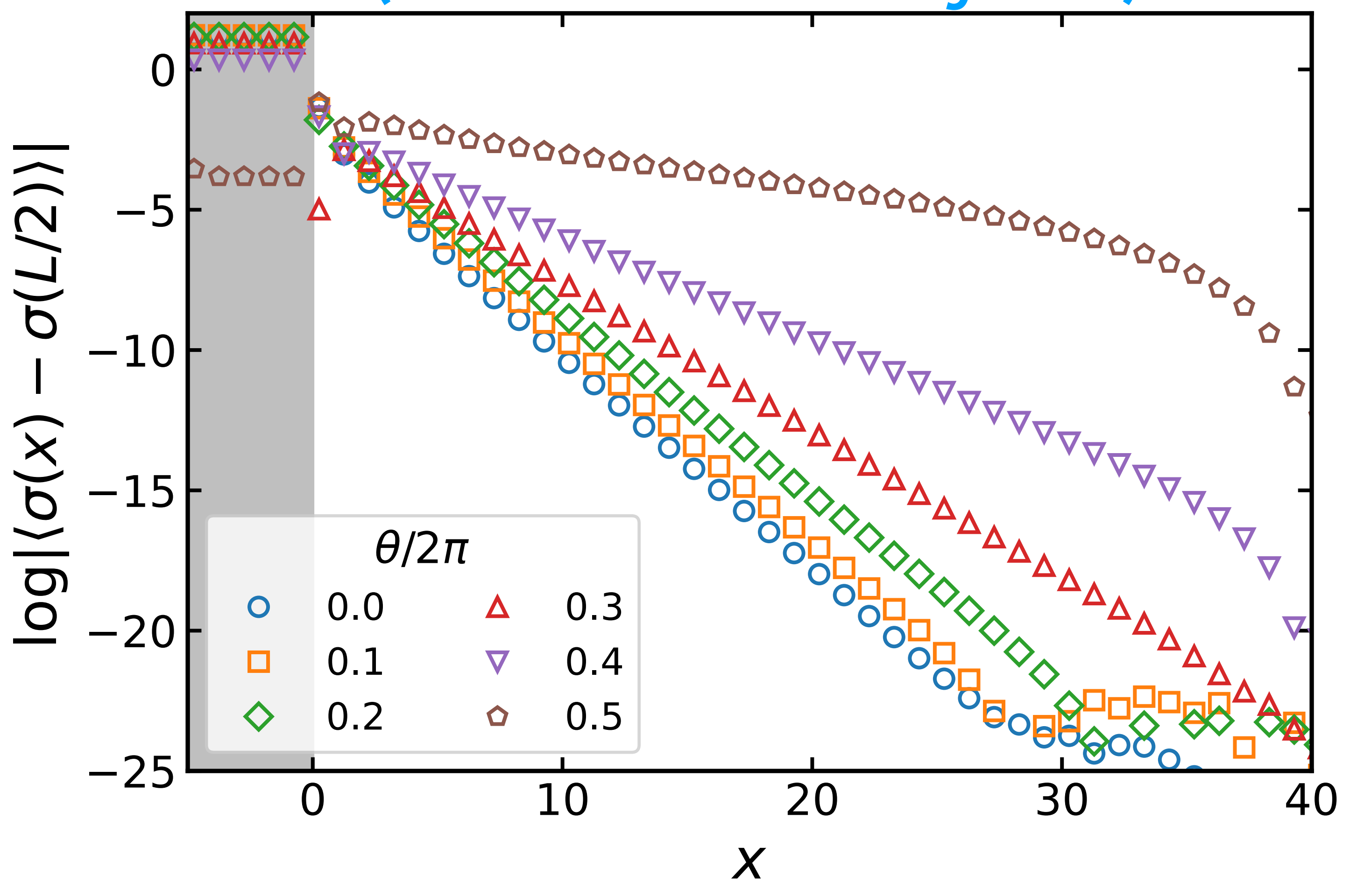
  we fit the effective mass by $M + \dfrac{\Delta M}{1 + Ce^{\Delta M x}}$ to obtain M

$N = 320,\ a = 0.25$

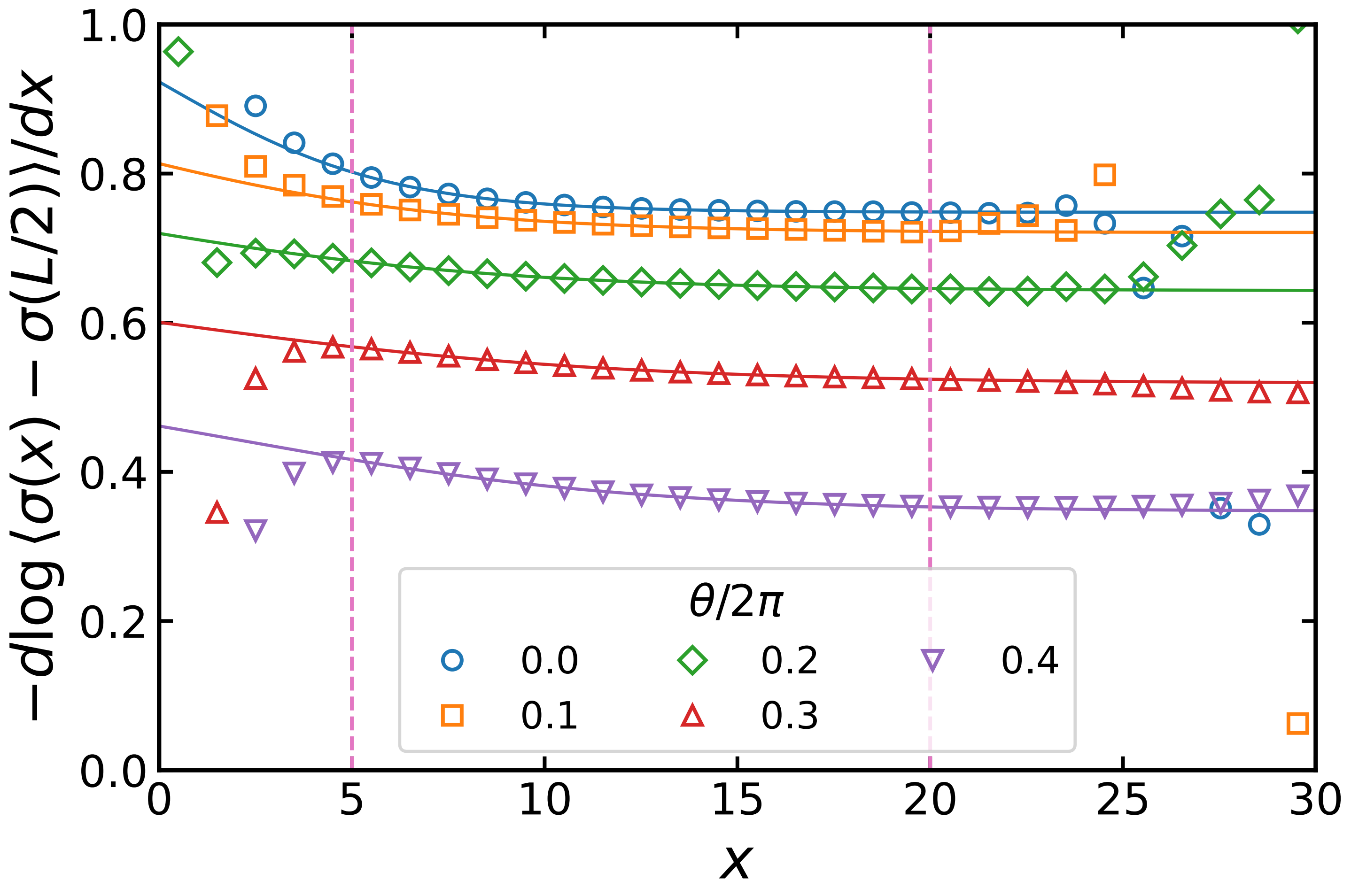
$m = 0.1,\ m_0 = 10$

sigma meson

(stable at any θ)

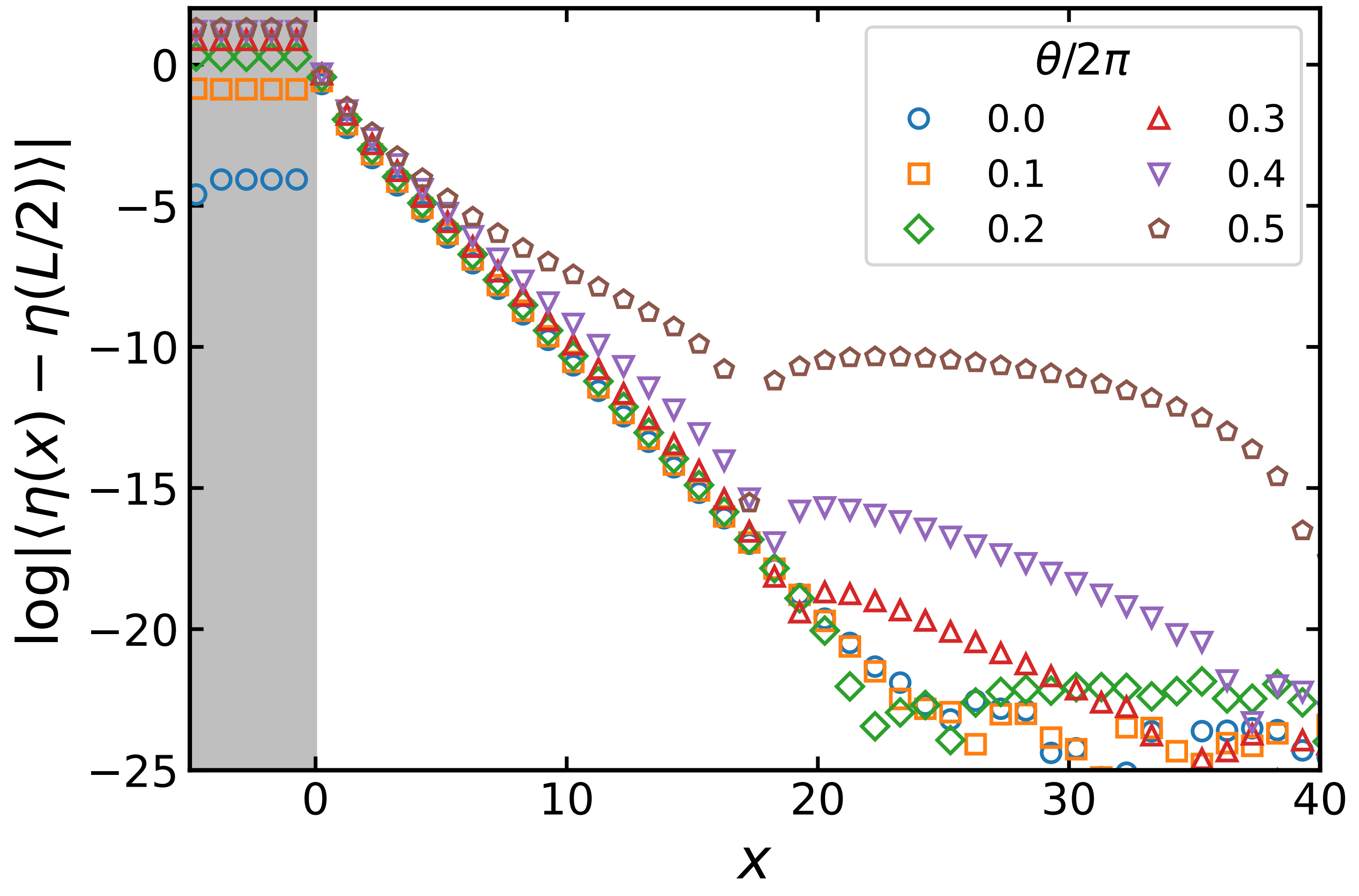


effective mass



eta meson

(unstable at $\theta \neq 0$)



Result of pion

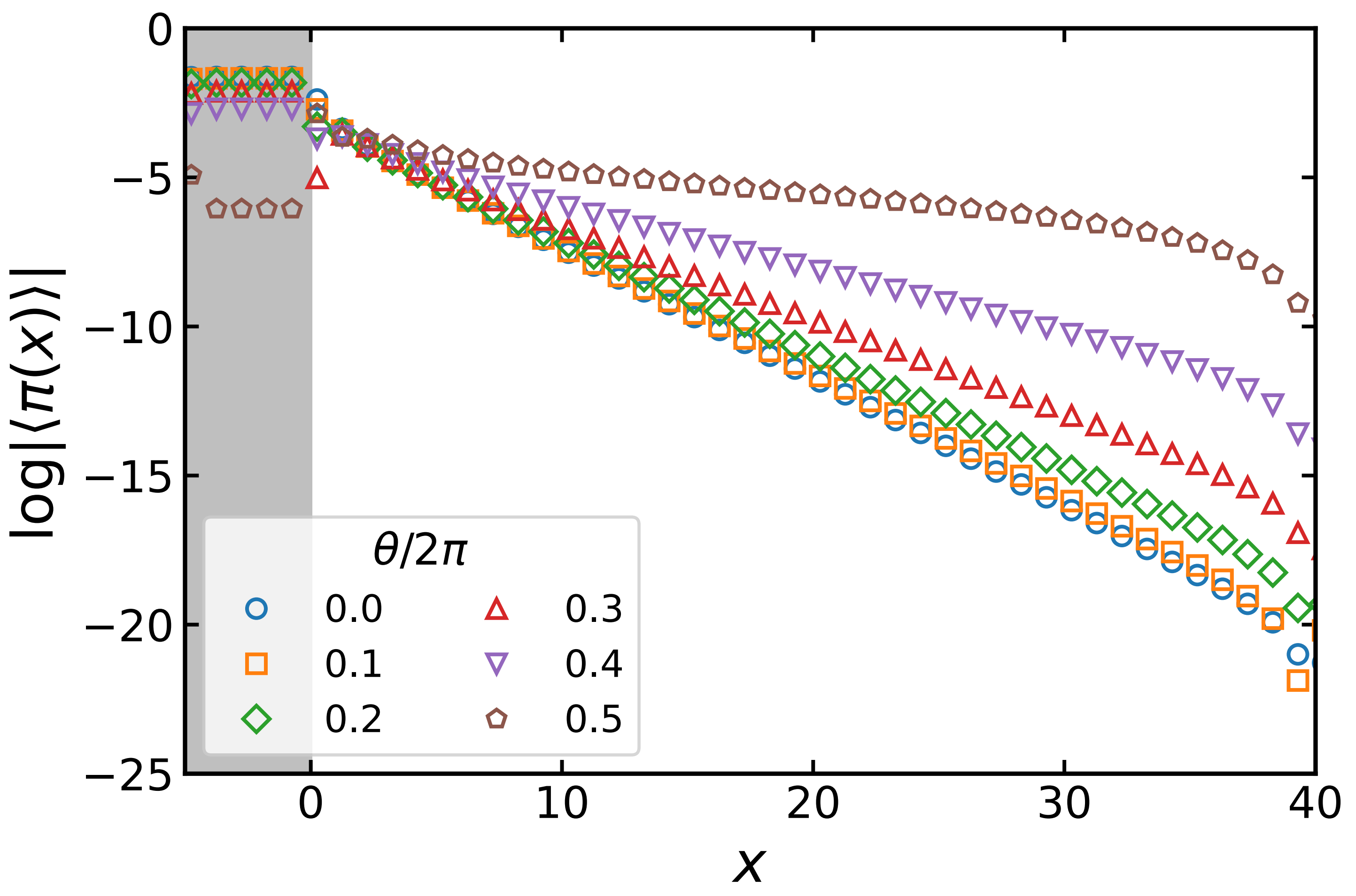
⚠️ $\langle \pi(x) \rangle = 0$ for the Dirichlet b.c. since such a boundary is isospin singlet

- We apply a flavor-asymmetric twist $m_{\mathrm{wings}} = m_0 \exp(\pm i \Delta \gamma^5)$ in the wings to induce the isospin-breaking effect

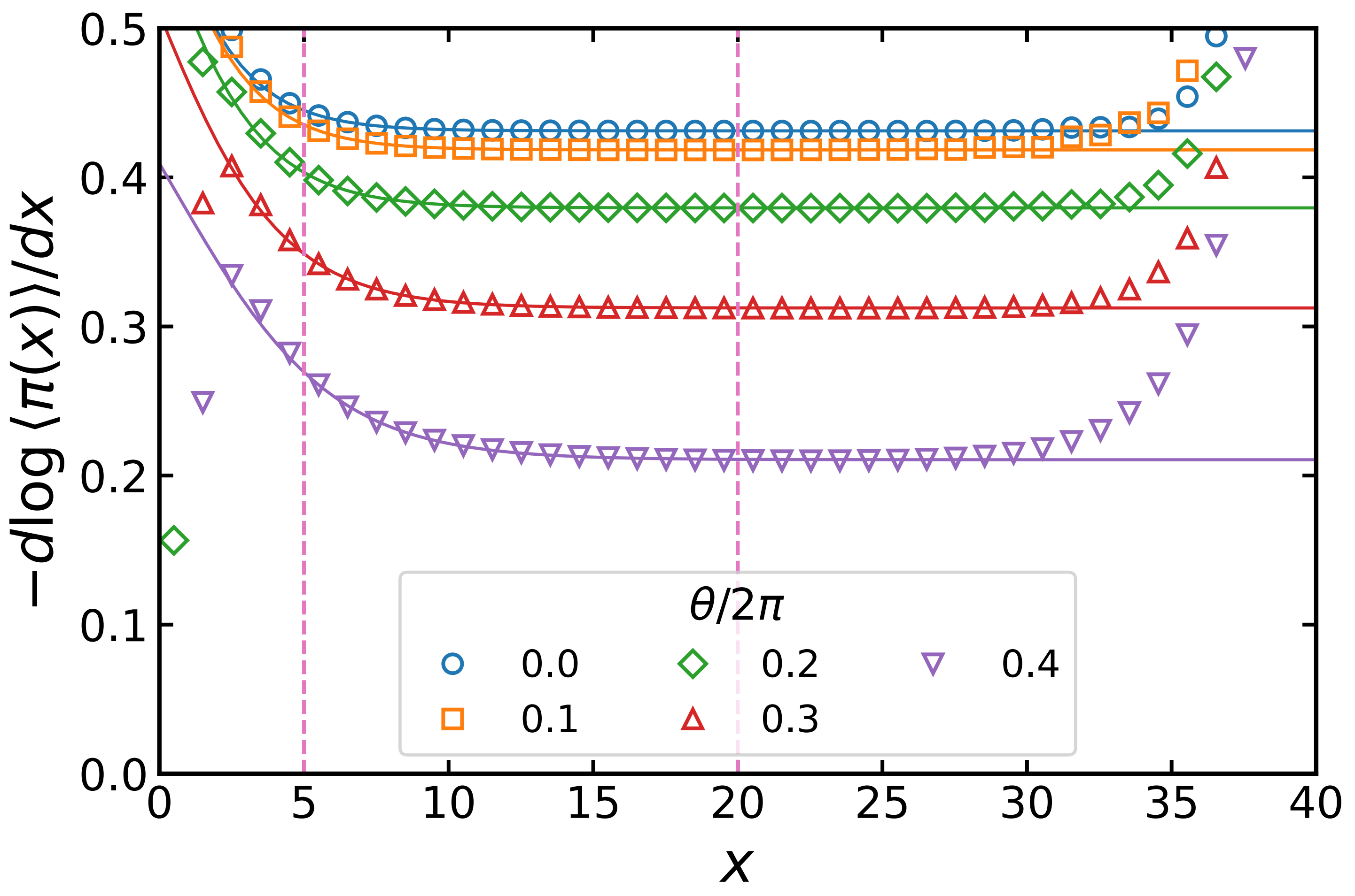
$N = 320,\ a = 0.25$

$m = 0.1,\ m_0 = 10,\ \Delta = 0.1$

pion

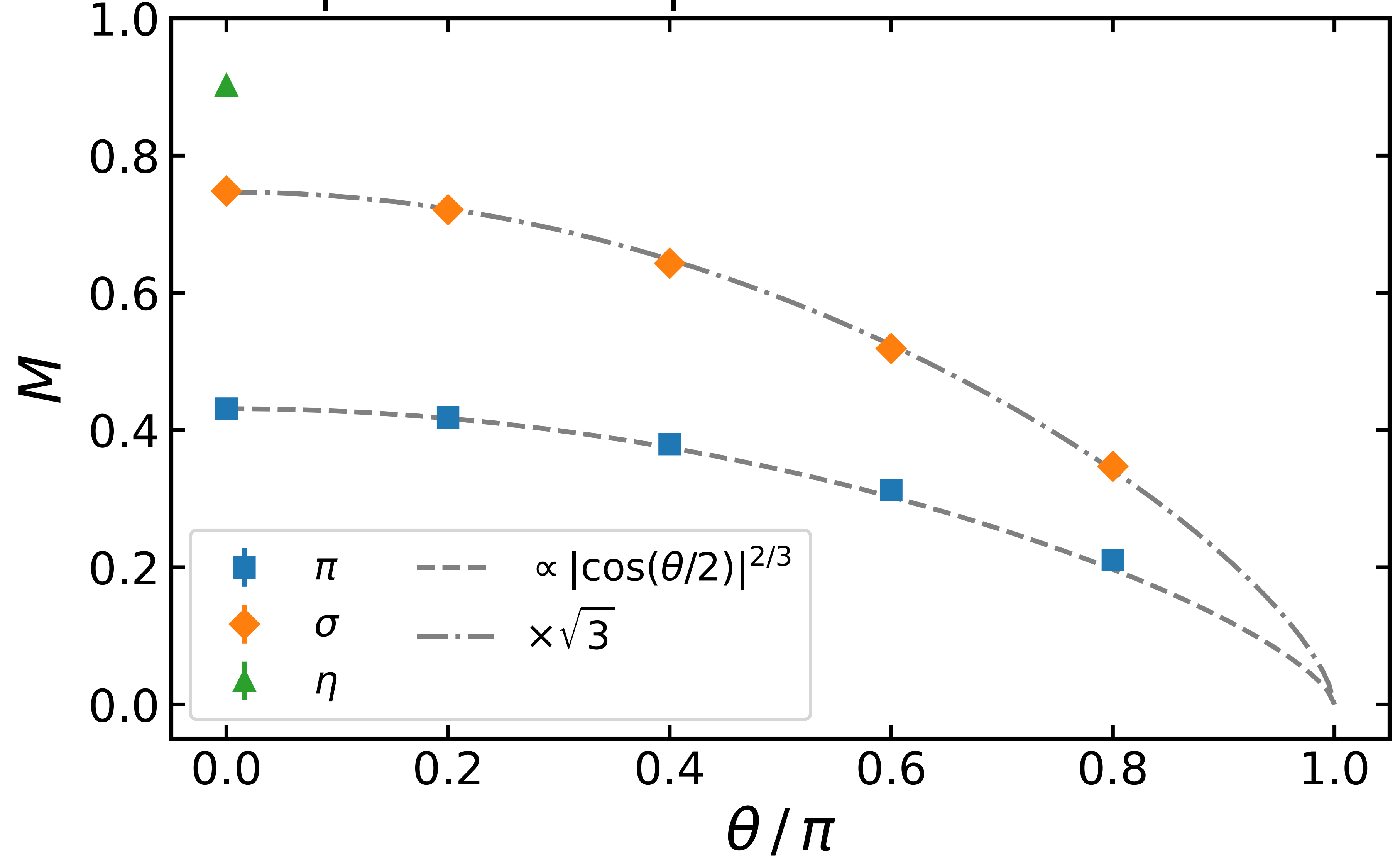


effective mass



summary

improved one-point-function scheme



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Dispersion-relation scheme

Obtain the dispersion relation $E = \sqrt{K^2 + M^2}$ directly from the excited states (momentum excitations of the mesons)

- ℓ-th excited state $|\Psi_\ell\rangle$

  = the lowest energy eigenstate satisfying $\langle \Psi_{\ell'} | \Psi_\ell \rangle = 0$ for $\ell' = 0, 1, \cdots, \ell - 1$

- obtained by DMRG, adding the projection term to H

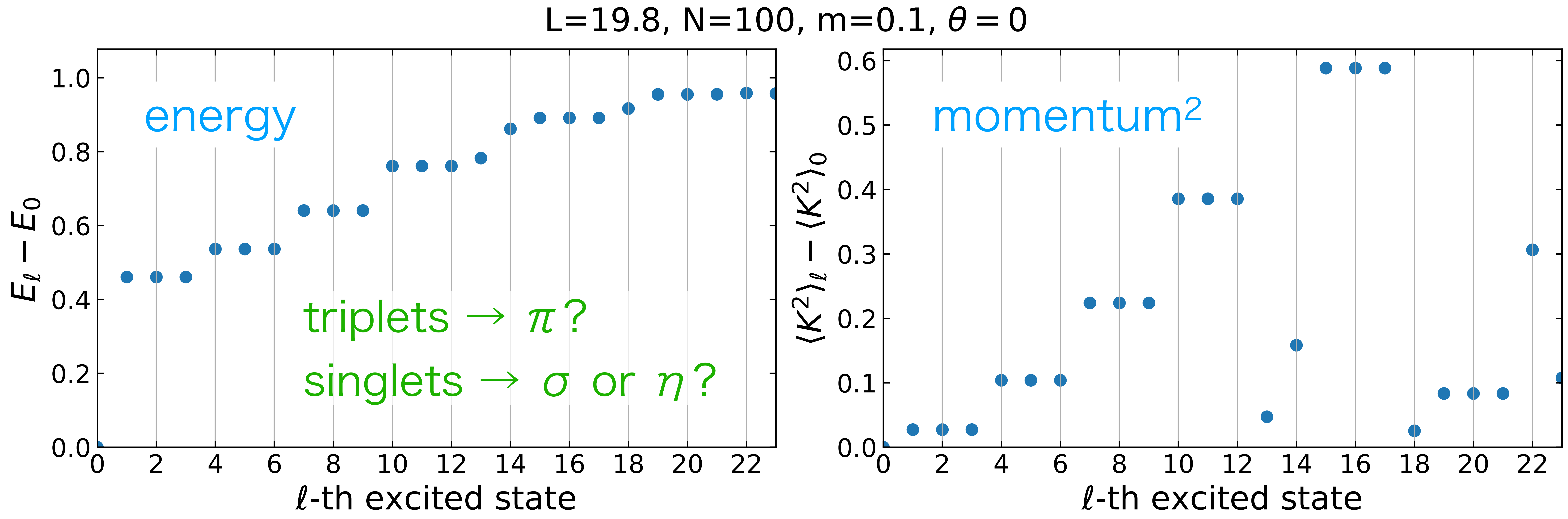
[Stoudenmire & White (2012)]
[Banuls et al. (2013)]

$$H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| \quad \text{—> cost function: } \langle \Psi_\ell | H | \Psi_\ell \rangle + W \sum_{\ell'=0}^{\ell-1} \left| \langle \Psi_{\ell'} | \Psi_\ell \rangle \right|^2 \qquad W > 0$$

- measure the energy E and the total momentum $K = \sum_f \int dx\, \psi_f^\dagger (i\partial_x - A_1)\psi_f$

Energy spectrum at $\theta = 0$

- energy gap: $\Delta E_\ell = E_\ell - E_0$
- momentum square: $\Delta K_\ell^2 = \langle K^2 \rangle_\ell - \langle K^2 \rangle_0$
- identify the states by measuring quantum numbers: $\mathbf{J}^2$, J_z, $G = Ce^{i\pi J_y}$



Result of quantum numbers

- triplets: $\mathbf{J}^2 = 2, \;\; J_z = (0, \pm 1), \;\; G > 0$

  —> pion ($J^{PG} = 1^{-+}$)

- singlets: $\mathbf{J}^2 = 0, \;\; J_z = 0,$

  $G > 0$ ($\ell = 13, 14, 22$) —> sigma meson ($J^{PG} = 0^{++}$)

  $G < 0$ ($\ell = 18, 23$) —> eta meson ($J^{PG} = 0^{--}$)

triplets

ℓ	$\boldsymbol{J}^2$	J_z	G
1	2.00000004	0.99999997	0.27872443
2	2.00000012	-0.00000000	0.27872416
3	2.00000004	-0.99999996	0.27872443
4	2.00000007	0.99999999	0.27736066
5	2.00000006	0.00000000	0.27736104
6	2.00000009	-0.99999998	0.27736066
7	2.00000010	1.00000000	0.27536687
8	2.00000002	0.00000000	0.27536702
9	2.00000007	-0.99999998	0.27536687
10	2.00000007	0.99999998	0.27356274
11	2.00000005	0.00000001	0.27356277
12	2.00000007	-0.99999999	0.27356274
15	1.99999942	0.99999966	0.27173470
16	2.00000052	0.00000000	0.27173482
17	2.00000015	-1.00000003	0.27173470
19	2.00009067	1.00004377	0.27717104
20	2.00002578	-0.00000004	0.27717020
21	2.00003465	-1.00001622	0.27717104

singlets

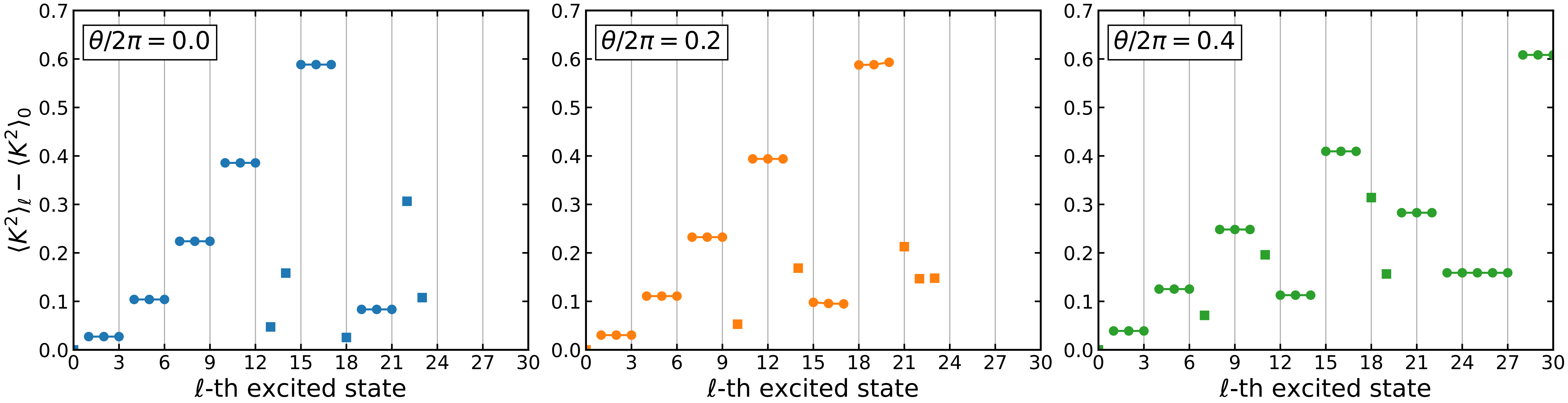
ℓ	$\boldsymbol{J}^2$	J_z	G
0	0.00000003	-0.00000000	0.27984227
13	0.00000003	0.00000000	0.27865844
14	0.00000003	0.00000000	0.27508176
18	0.00000028	0.00000006	-0.27390909
22	0.00001537	0.00000115	0.26678987
23	0.00003607	-0.00000482	-0.27664779

Extension to $\theta \neq 0$

- G-parity is no longer the quantum number —> η disappears
- singlet projection to obtain σ with reasonable computational cost

$$H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| + W_J \mathbf{J}^2$$

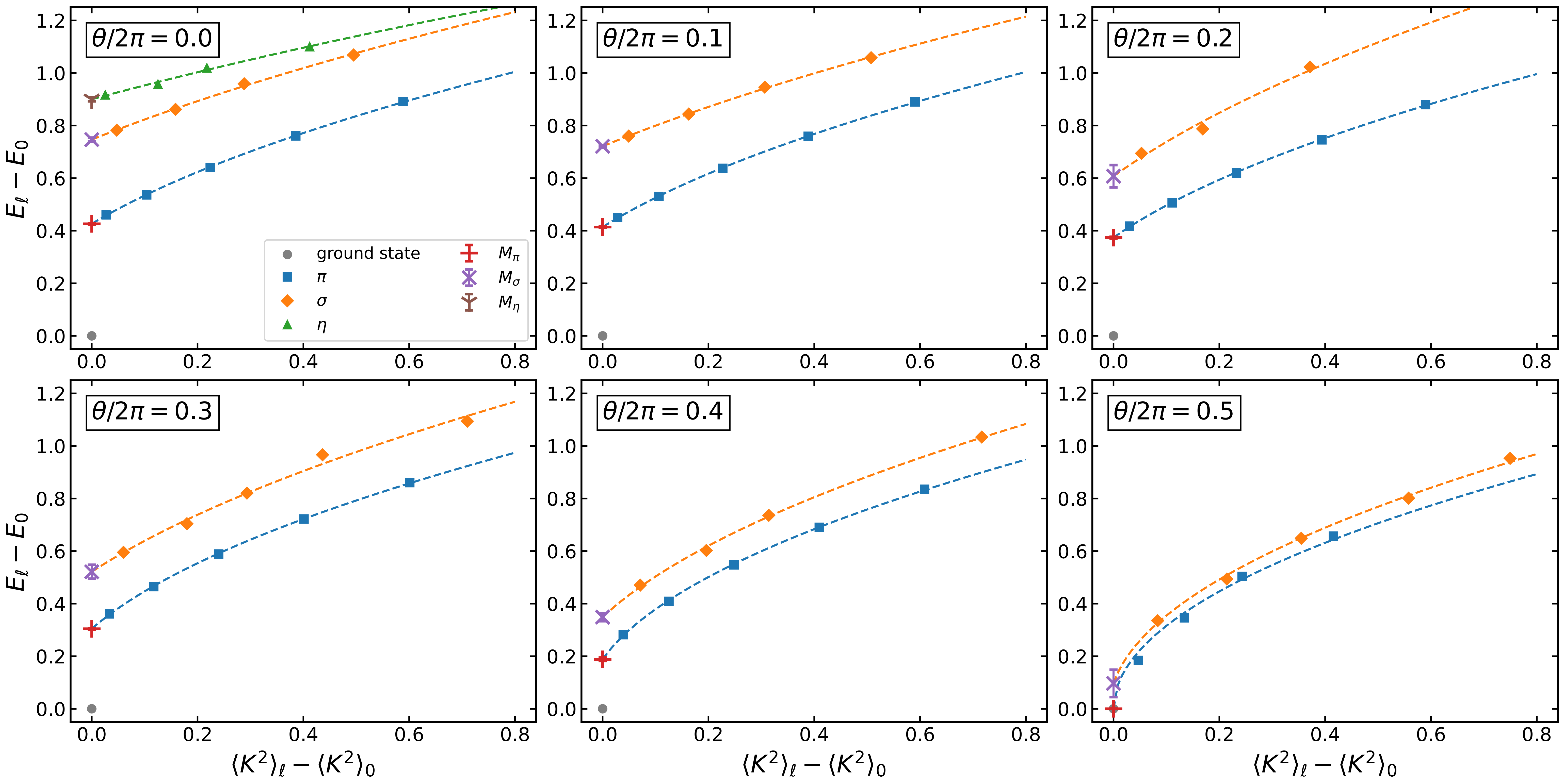
momentum^2



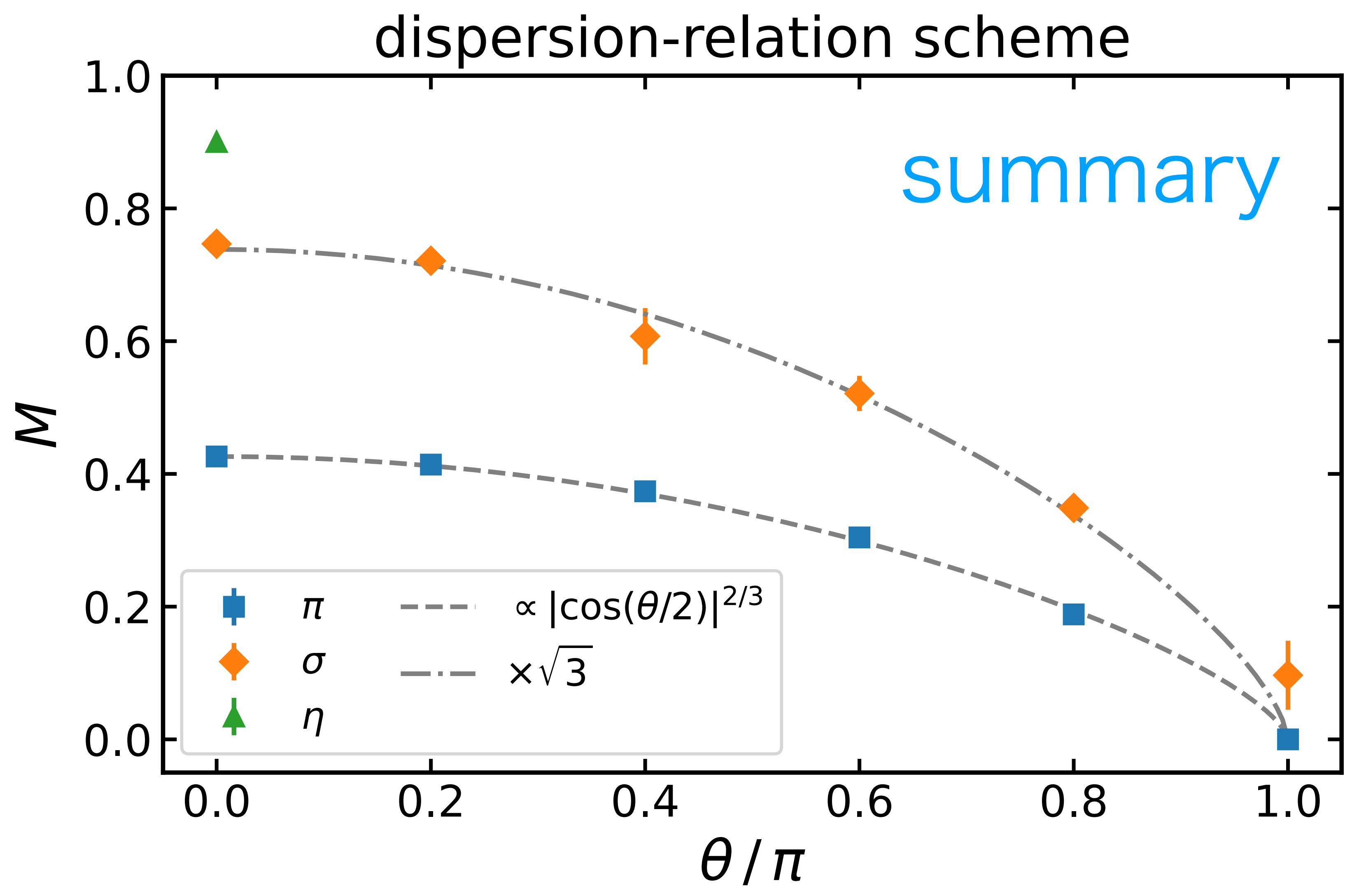
Result of dispersion relation

- plot ΔE_ℓ against ΔK_ℓ^2 and fit the data by $\Delta E = \sqrt{b^2 \Delta K^2 + M^2}$ for each meson

energy vs momentum^2



Around $\theta/2\pi = 0.2$, σ is contaminated by a remnant of η due to the mixing



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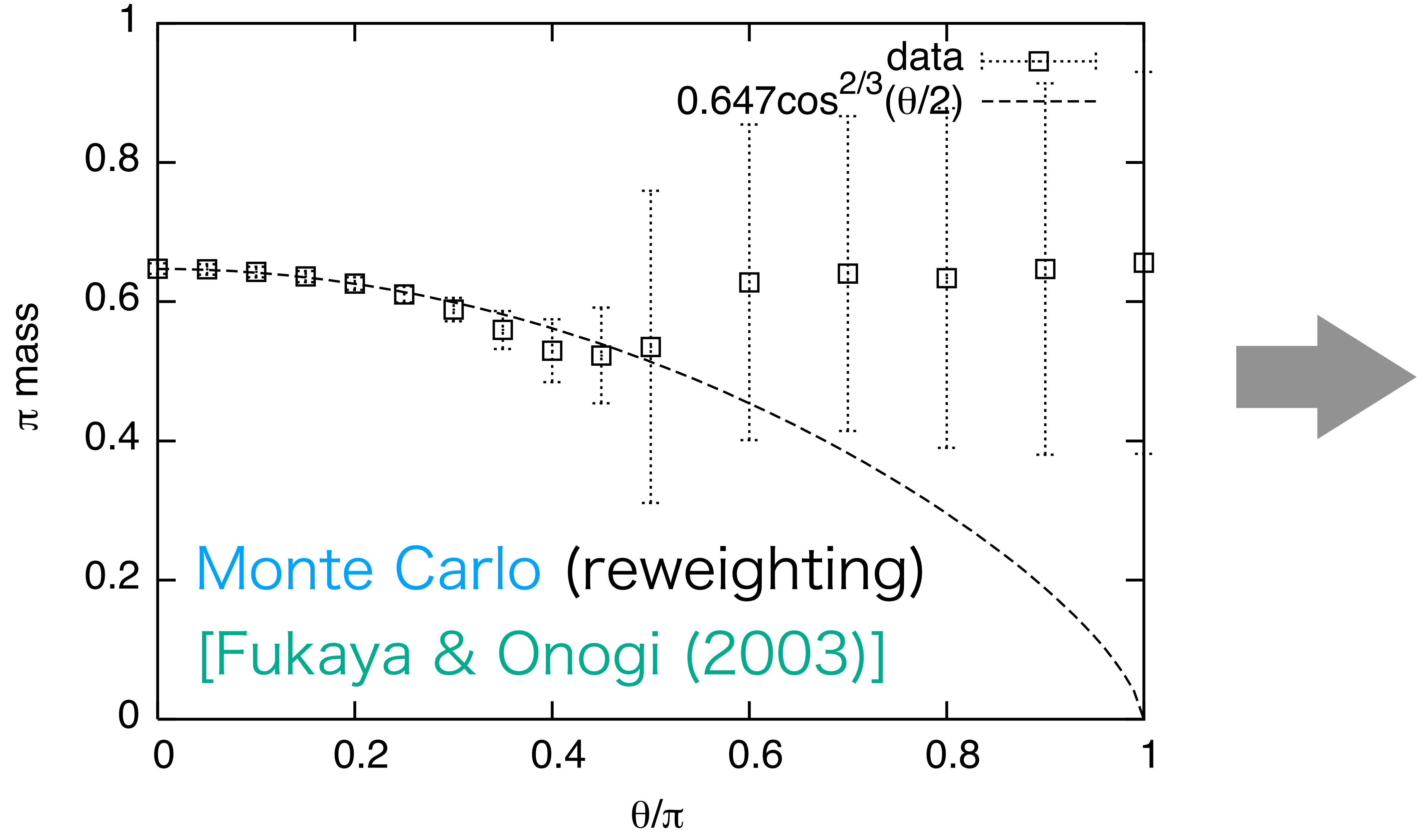
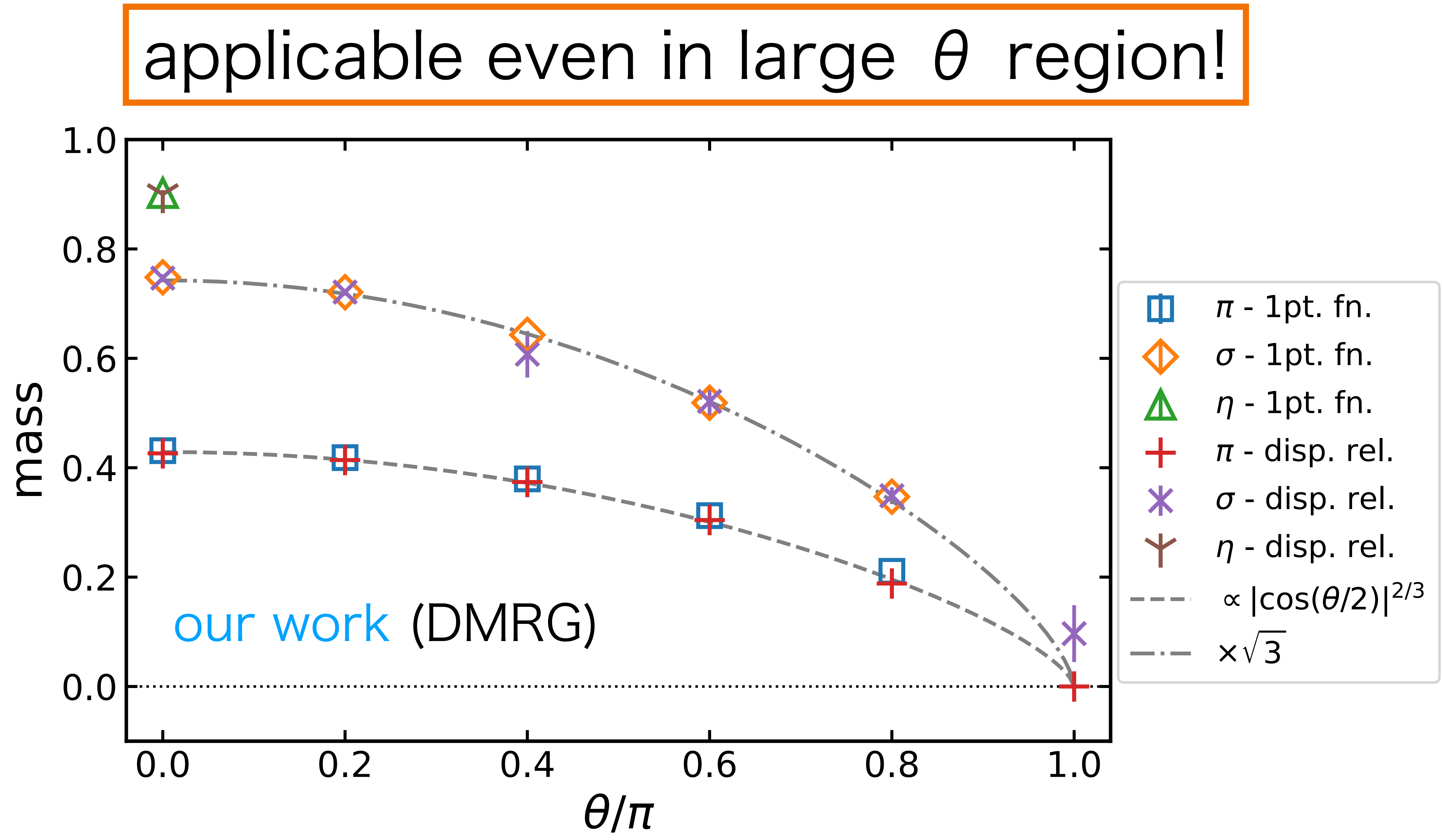
Summary

- The two schemes give consistent results and look promising
- consistent with predictions by the bosonization [Coleman (1976)] [Dashen et al. (1975)]

$$M_\pi(\theta) \propto |\cos(\theta/2)|^{2/3} \qquad M_\sigma(\theta)/M_\pi(\theta) = \sqrt{3}$$





Future prospect

- Extension to 2+1 dimensions, where the gauge field is dynamical
- Application to the model with chemical potential:
  How the spectrum changes in the high-density region?
- Analyses using the wave functions of the excited states:
  scattering problem, entanglement property, etc.

Discussion

(1) correlation-function scheme

👍 generic method applicable to any case / off-diagonal elements

😥 sensitive to the bond dimension of MPS —> 😊 quantum computer?

(2) (improved) one-point-function scheme

👍 NOT sensitive to the bond dimension / easy to compute

😥 only the lowest state of the same quantum number as the boundary

(3) dispersion-relation scheme

👍 obtain various states heuristically / directly see wave functions

😥 how to generate excited states efficiently?

Thank you for listening.

CFT-like behavior at $\theta = \pi$

<u>bond dim. of MPS grows up with N at $\theta = \pi \to$ gapless?</u>

cf.) bond dim. D bounds the entanglement entropy of MPS: $S_{\mathrm{EE}} \lesssim \log D$

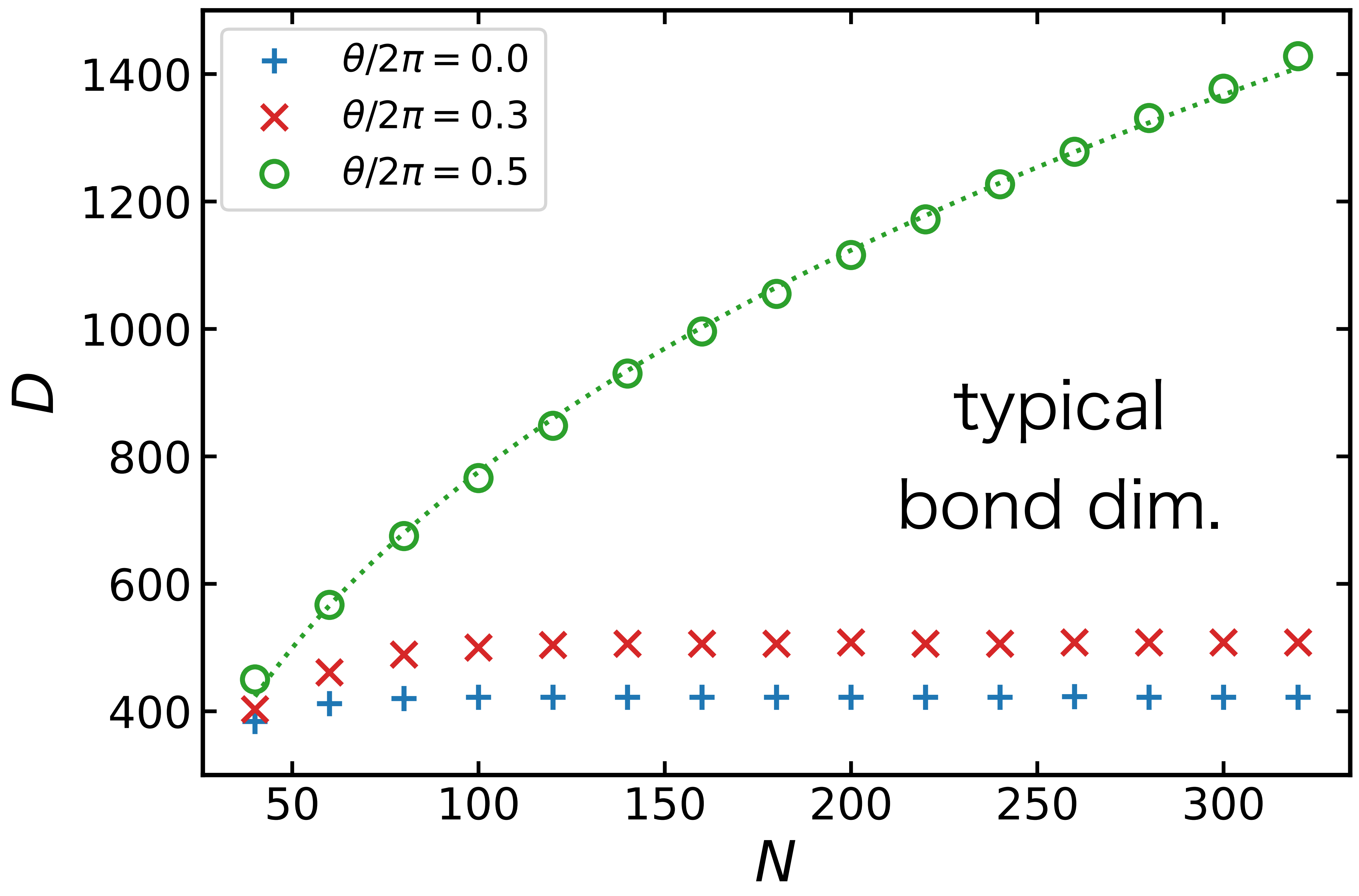
1+1d **gapped** : $S_{\mathrm{EE}} \sim \mathrm{const.}$

—> D is independent of the size N

1+1d **gapless** : $S_{\mathrm{EE}} \sim \frac{c}{3} \log N$

—> increases by power $D \sim N^{c/3}$

- central charge $c = 1$ in this case
  (deviation due to the finite a exists)



CFT-like behavior at $\theta = \pi$

- At $\theta = \pi$, the mass gap is exponentially small $\sim e^{-\# g^2/m^2}$ [Coleman (1976)] [Dempsey et al. (2024)]

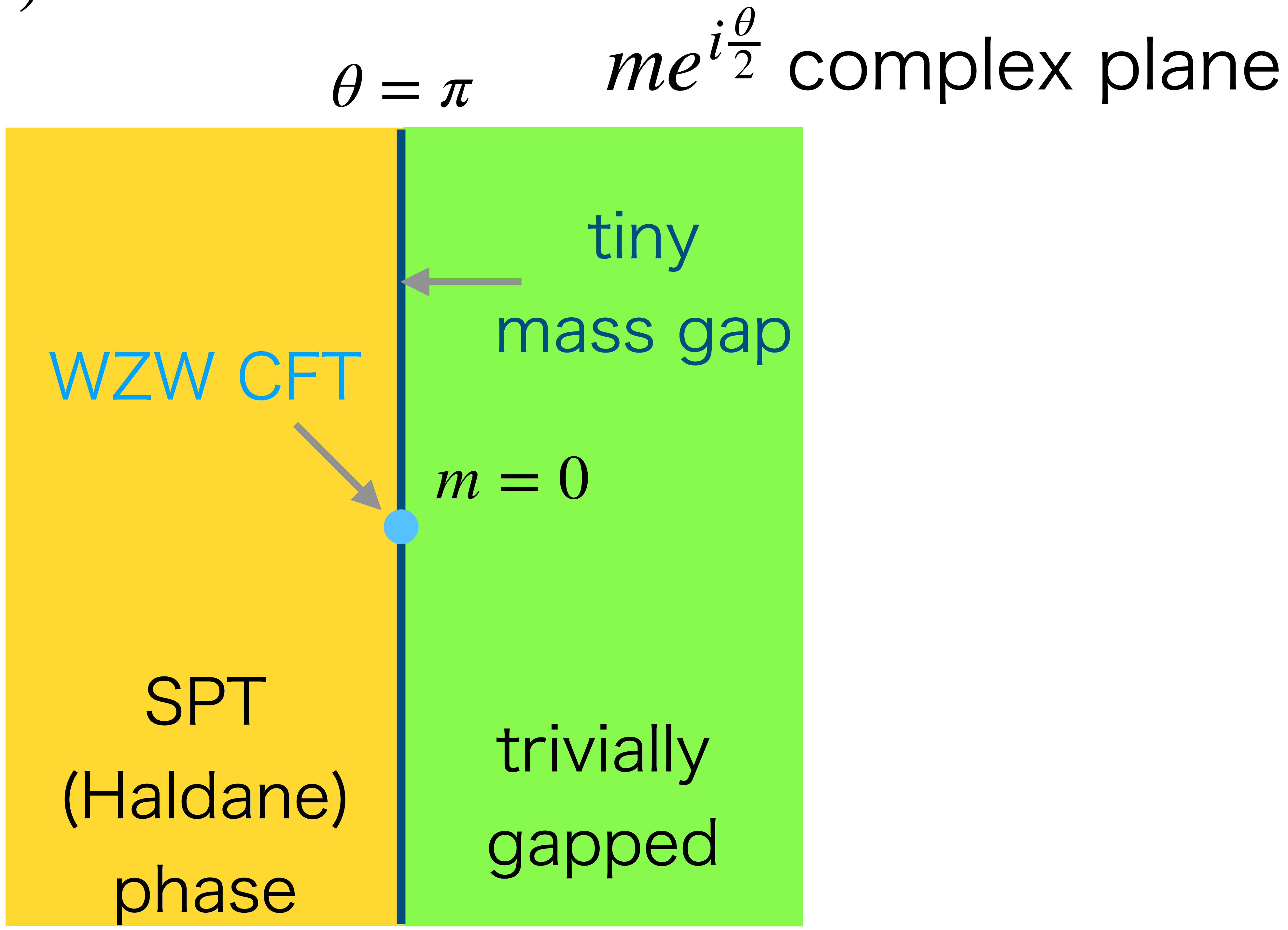
  cf.) SU(2)$_1$ WZW model with marginally relevant $J_L J_R$ deformation

- For the finite size L, the energy scale below $O(1/L)$ is invisible

  —> system is CFT-like if $L < e^{\# g^2/m^2}/g$

  our setup : $L = 80,\ m = 0.1,\ g = 1$

- compare the numerical result of 1pt. functions with the analytic calc. of WZW model

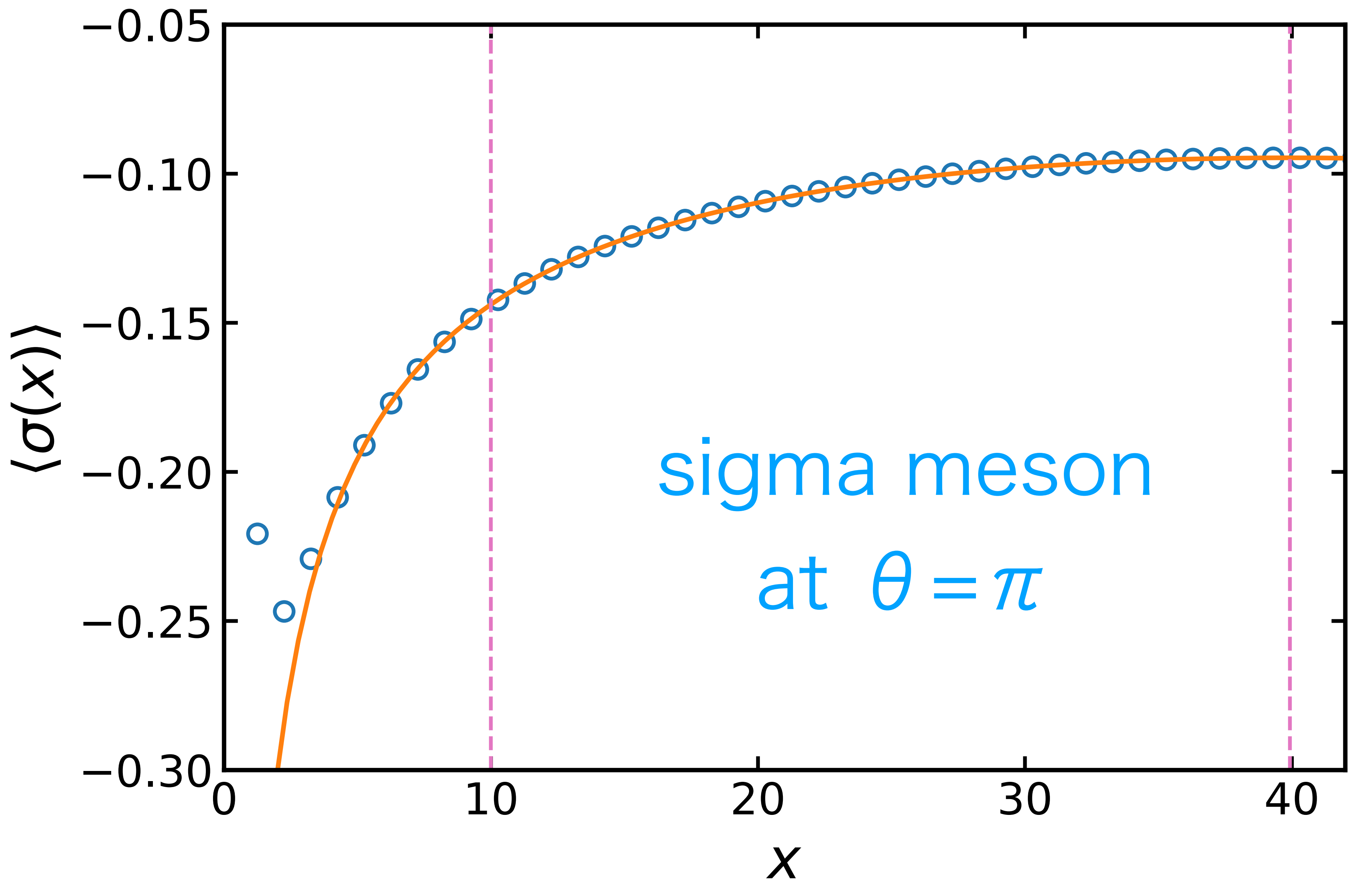


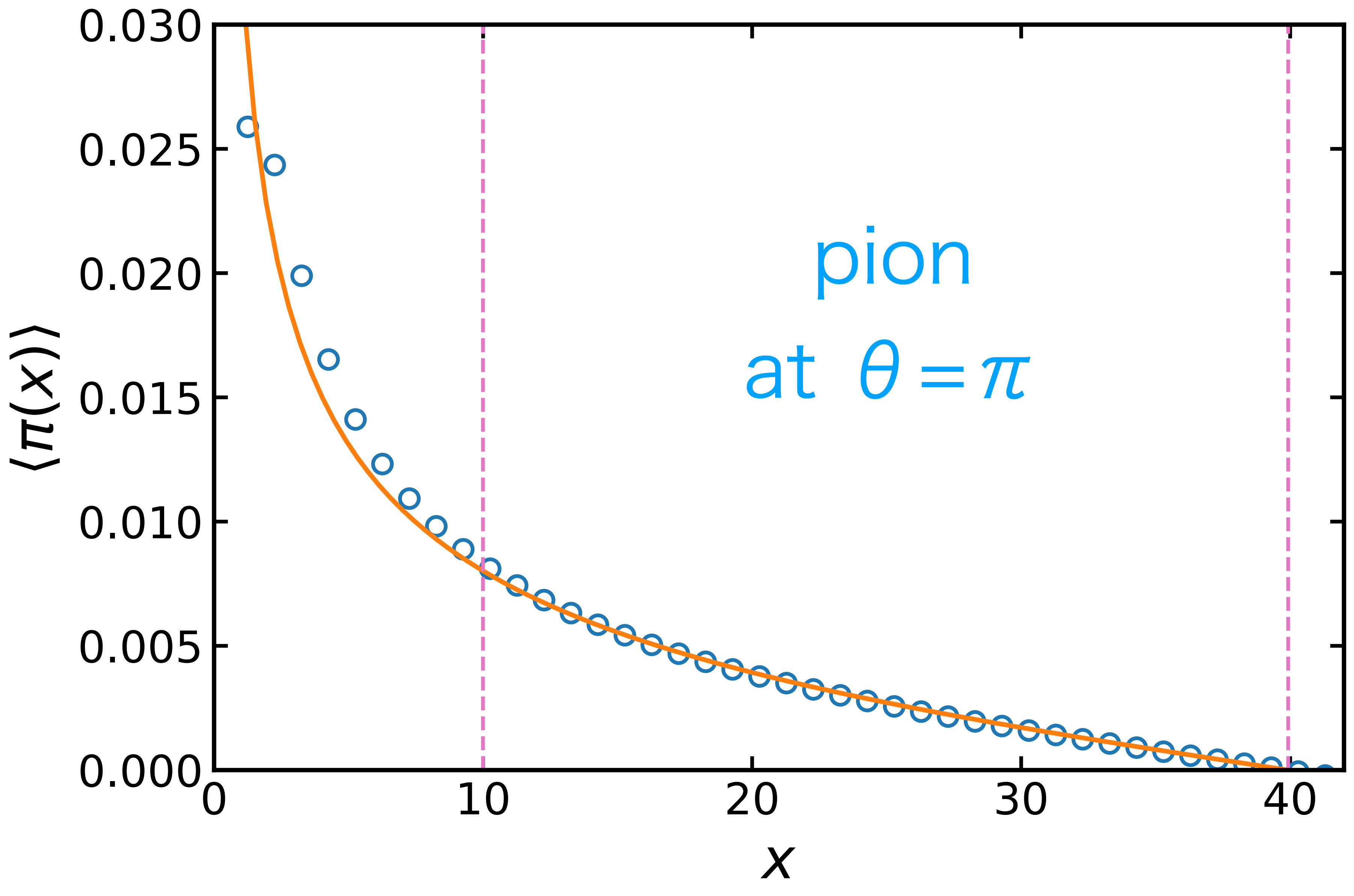
1pt. function of π and σ at $\theta = \pi$

- Dirichlet b.c. $\to \langle \sigma(x) \rangle \propto \dfrac{1}{\sqrt{\sin(\pi x/L)}}$
- isospin-breaking b.c. $\to \langle \pi(x) \rangle \propto \dfrac{\sin[\Delta(1 - 2x/L)]}{\sqrt{\sin(\pi x/L)}}$

mirror-image method

cf.) appendix A of JHEP09 (2024) 155

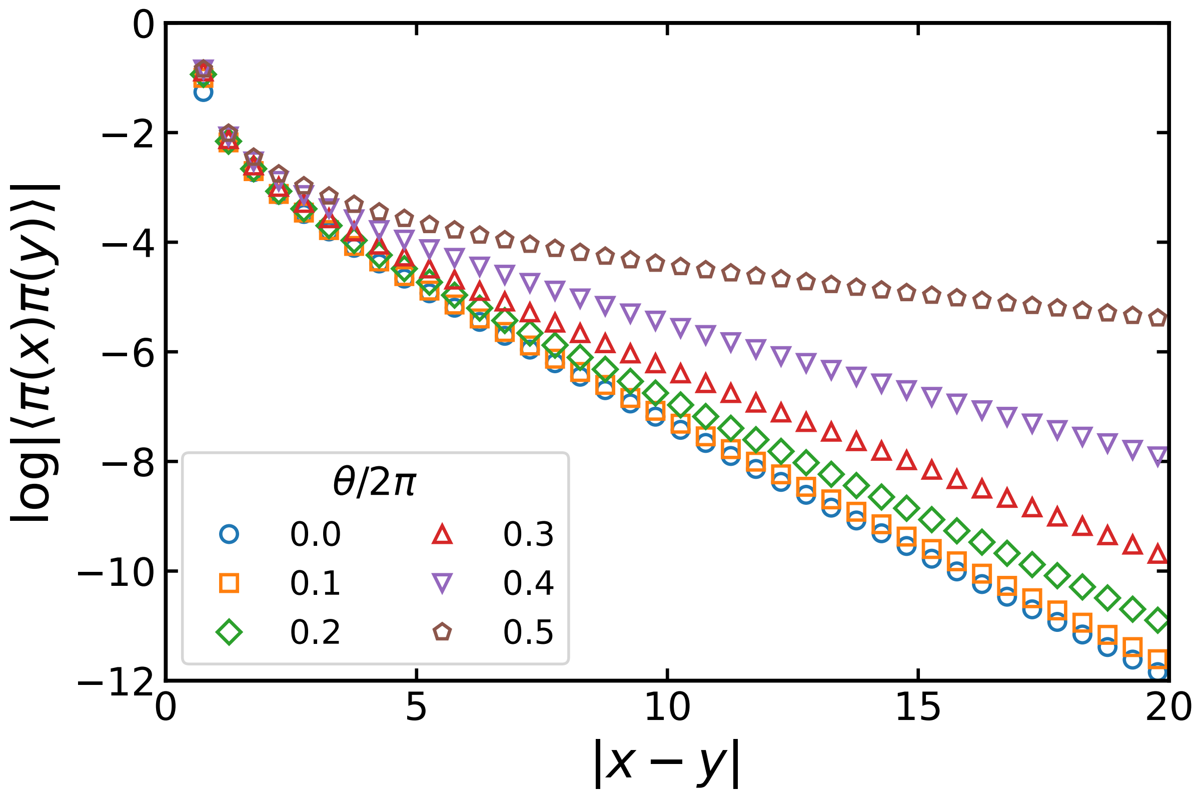


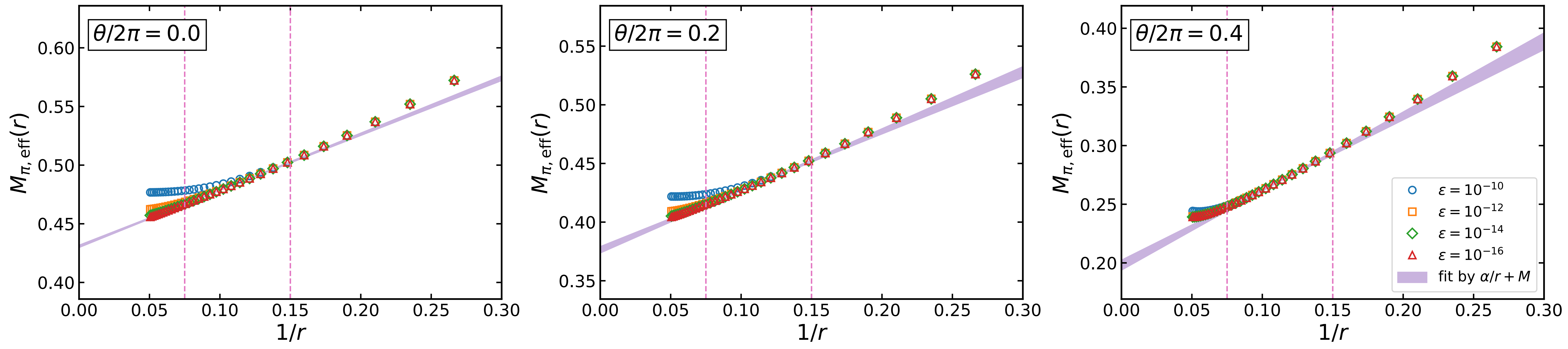


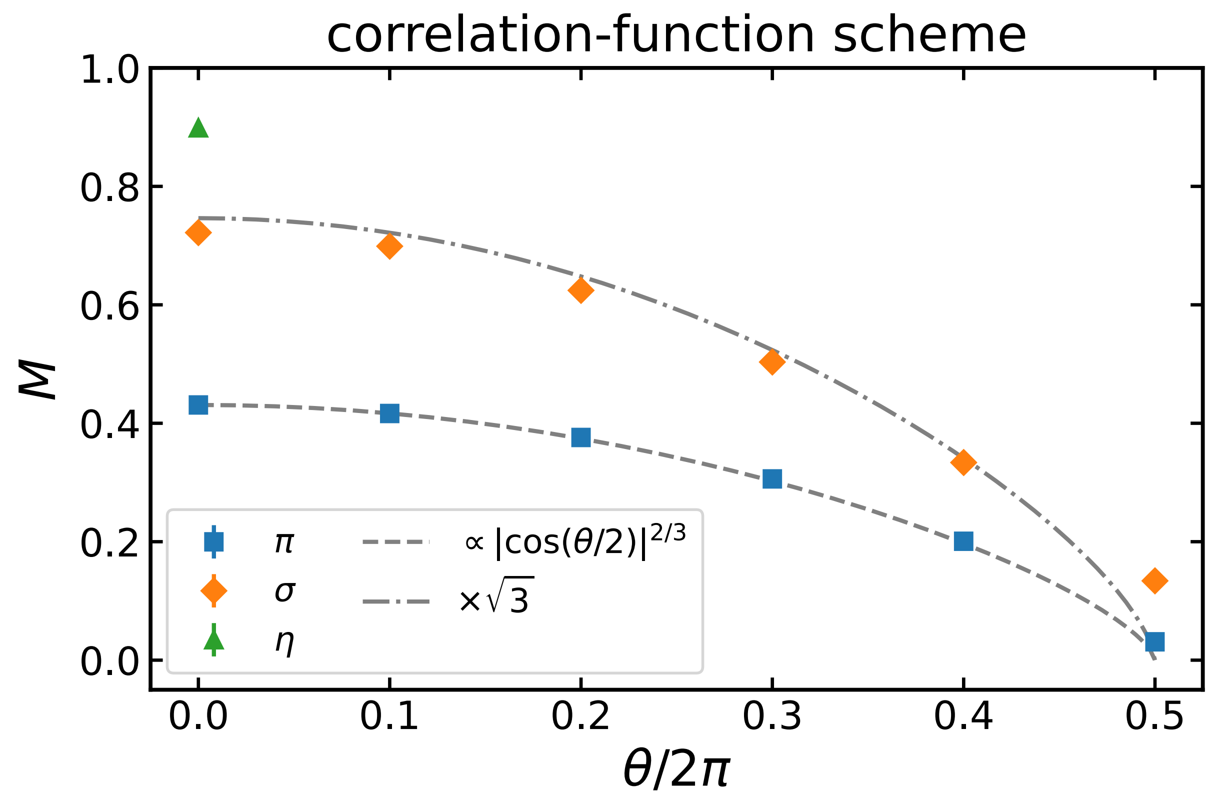
consistent with WZW model in the bulk

Correlation-function sheme

- spatial 2-point correlation function: $C_\pi(r) = \langle \pi(x)\pi(y) \rangle \sim \frac{1}{r^\alpha} e^{-Mr}$ $\quad r = |x - y|$
- effective mass: $M_{\pi,\mathrm{eff}}(r) = -\frac{d}{dr} \log C_\pi(r) \sim \frac{\alpha}{r} + M$
- 1/r behavior is observed only when the bond dim. is large
- mass is given by $r \to \infty$ extrapolation





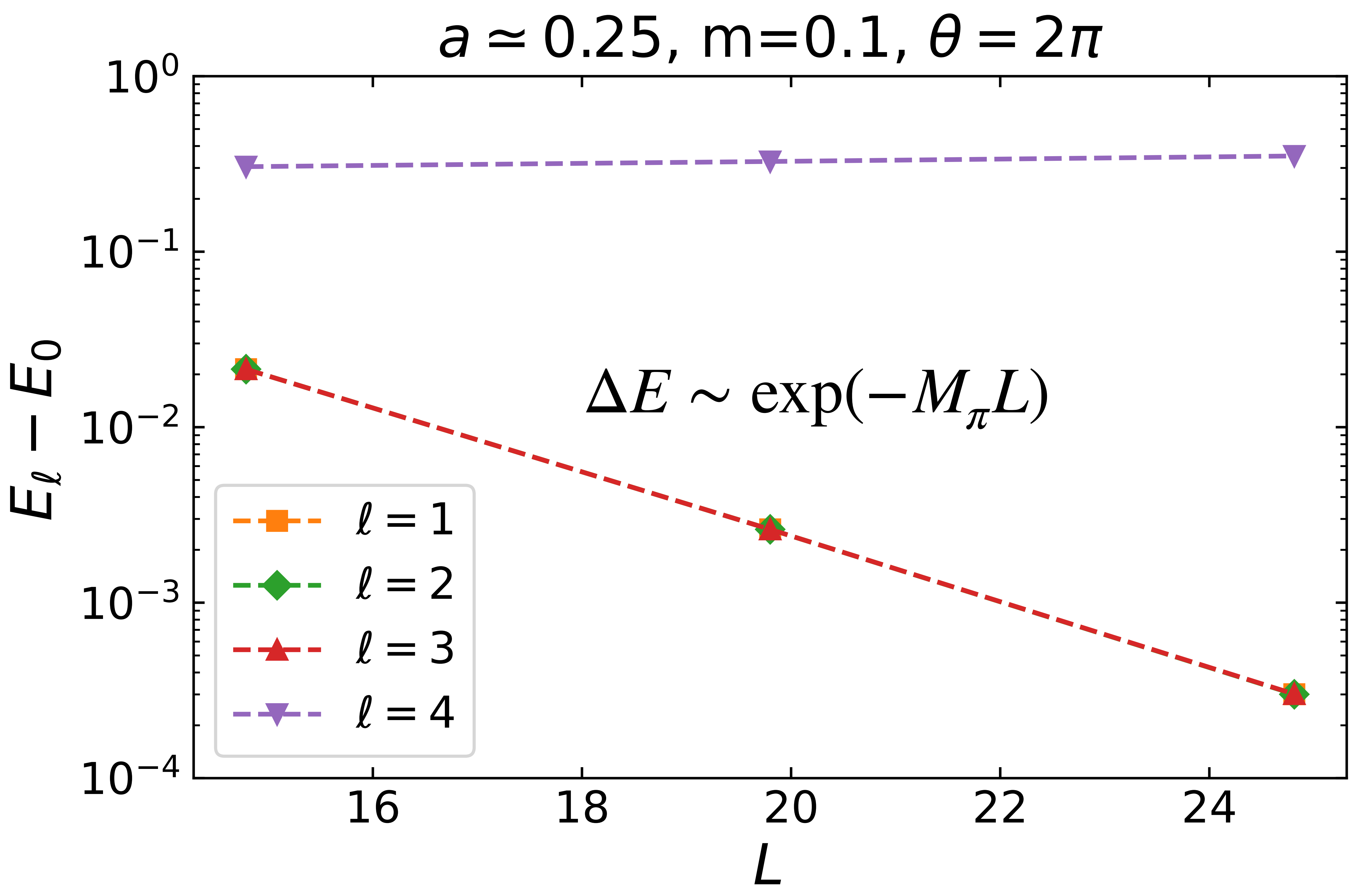


Degeneracy of the ground states

- one ground state + three 1st excited states are observed by DMRG at $\theta = 2\pi$.
- energy gap $\sim \exp(-M_\pi L) \to 0$
- solve $\Delta E_\ell = C_0 + \exp(-ML + C_1)$ for $\ell = 1$;

  $M = 0.41767,\ C_0 = -0.00002,\ C_1 = 2.33326$
- cf.) $M_\pi = 0.4175(9)$ by 1pt-fn. scheme
- DMRG is hard when L is small or $\theta \to \pi+$

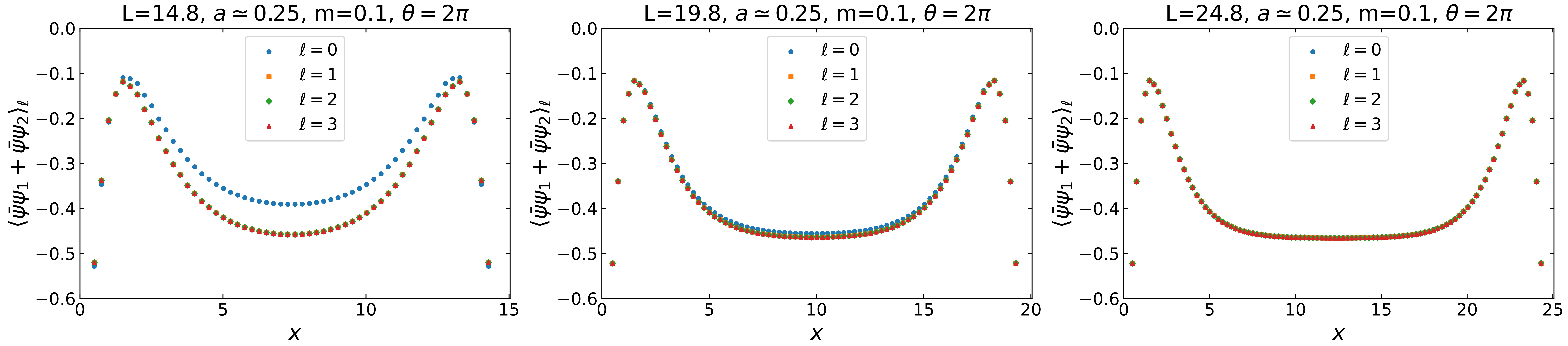
energy gap of the ℓ-th excited state



Local observables

- local scalar condensate $\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$ (isospin singlet) at $\theta = 2\pi$
- degeneracy in $L \to \infty$

small L → large L

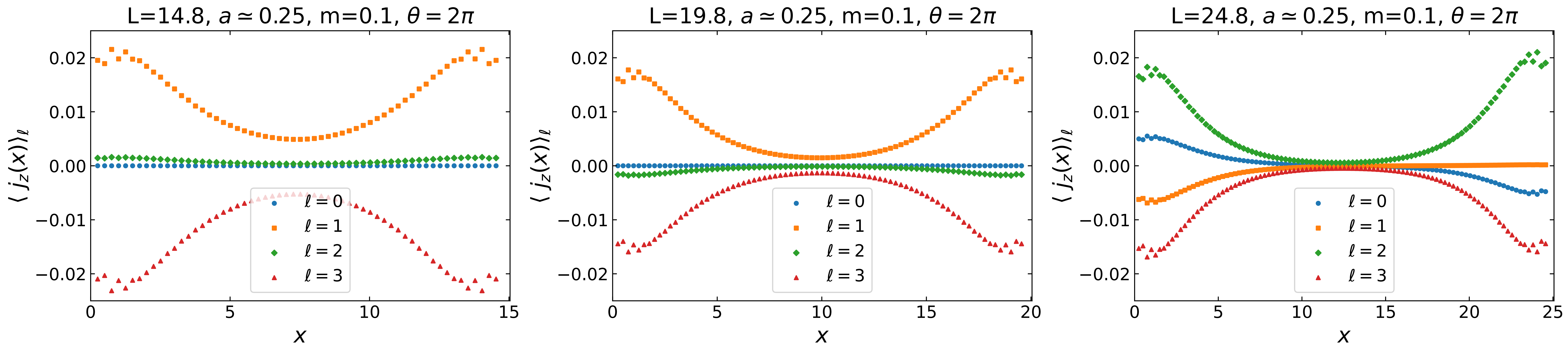


Local isospin

- local isospin $j_z(x) = \frac{1}{2}(\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2)$ at $\theta = 2\pi$
- finite L : singlet + triplet —> $L \to \infty$: doublet × doublet

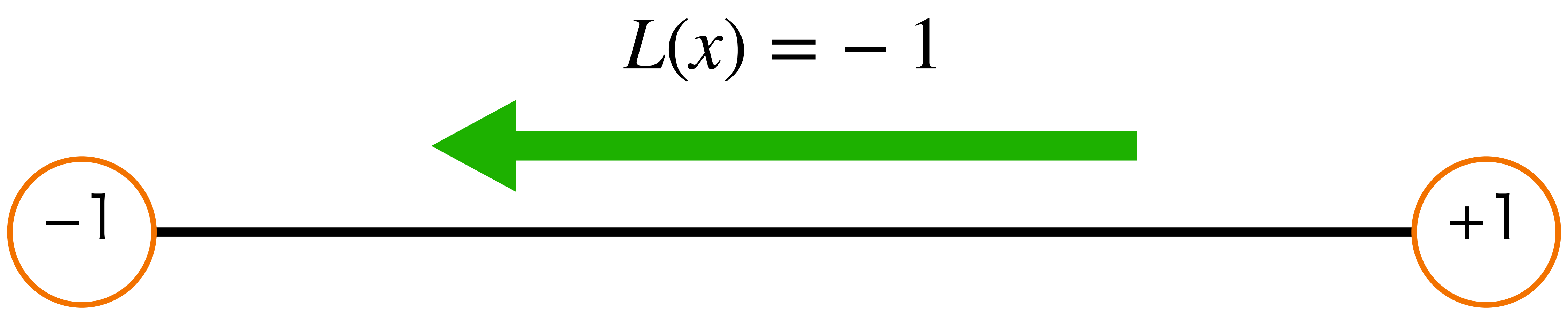
  interaction is suppressed exponentially and the edge modes are decoupled

small L → large L



Electric charge and electric field

- charge density: $\rho(x) = \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2$
- induced electric field: $L(x) = \int_0^x dy\, \rho(y)$



cancel the background electric field

$E = \theta/2\pi = +1$ from θ term