The construction and dependence of the initial tensor for the TRG with the Steiner tree problem.

[K. N. arXiv:2307.14191]
[K. N., M.Schneider arXiv:2407.14226]

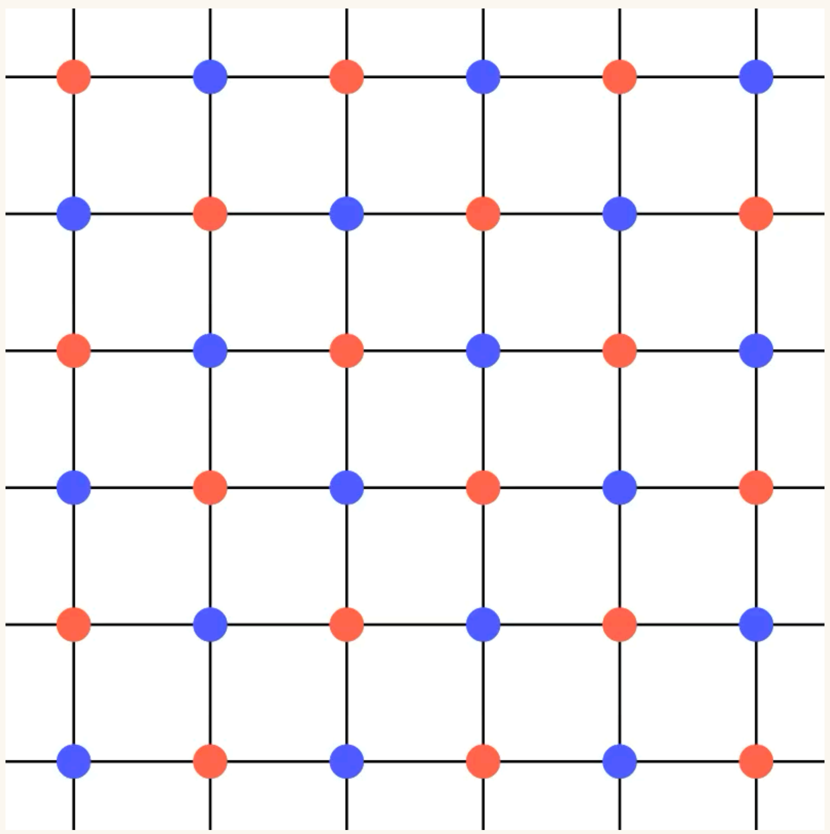
Katsumasa Nakayama (RIKEN)
Manuel Schneider (NYCU)

2024/11/15@Kanazawa, Japan.

◉ Tensor renormalization group (TRG)

◇ What is the TRG?

→ TRG is approximated contraction of (locally connected) Tensor network.



[https://smorita.github.io/TN_animation/]

→ TRG require typical form of the tensor representation.

(Should be represented by graph)

$$Z = \mathrm{Tr} \sum_{i \in \mathrm{lattice}} A_{x_i y_i x'_i y'_i}$$

◇ What is the Tensor network?

→ Tensors are represented by each lattice points.

→ Indices are represented by line segments.

= Each index is only contained in two tensors.

Initial tensor network construction

Tensor network rep. of 2dim Ising

- How to find tensor network representation?

(e.g.): 2dim Ising model (Partition function)

Boltzmann factor

$$Z = \sum_{\sigma} \prod_{x,y} e^{\sigma_{x,y}\sigma_{x+1,y}+\sigma_{x,y}\sigma_{x,y+1}} = \sum_{\sigma} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}}$$

→ Tensors $K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}}$ do not construct tensor network.
(Index $\sigma_{x,y}$ is included in three tensors)

$$K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}}, K_{\sigma_{x-1,y}\sigma_{x,y}\sigma_{x-1,y+1}}, K_{\sigma_{x,y-1}\sigma_{x+1,y-1}\sigma_{x,y}}$$

↓

- Common method: (Taylor) expansion. and $\sigma^2 = 1$
or SVD and redefinition.

Tensor network rep. by delta matrix

[K. Nakayama, M.Schneider arXiv:2407.14226]

- Key point: Expansion produces tensor network rep. using property, e.g. $\sigma^2 = 1$ or decomposition

→ Problem: More complicated Boltzmann factor produces complicated form (difficult to find the tensor network).

- Our proposal: Index shift by delta matrix.

(e.g.): 2dim Ising model (periodic b.c.)

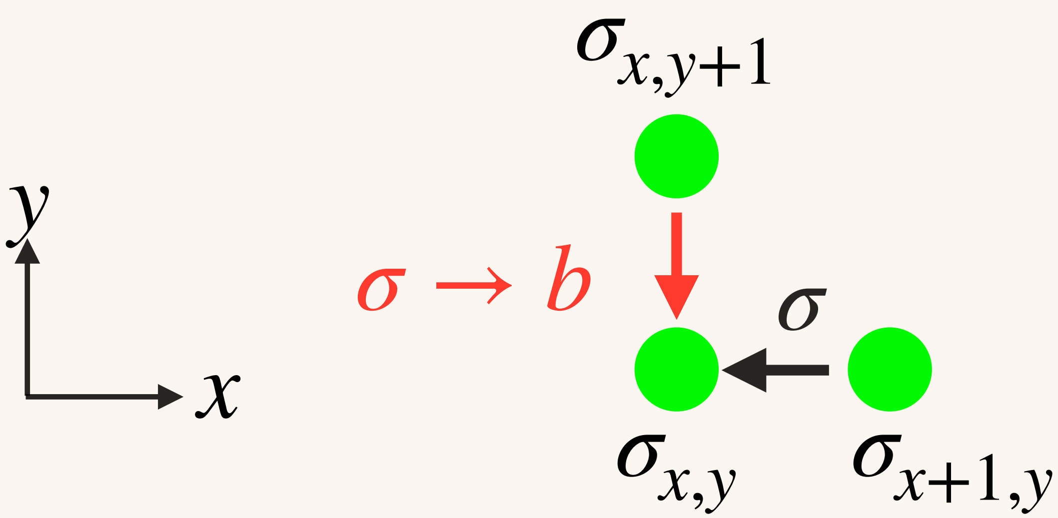
$$Z = \sum_{\sigma} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}} = \sum_{\sigma,b} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y+1}} \delta_{b_{x,y+1}\sigma_{x,y+1}} = \sum_{\sigma,b} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y+1}} \delta_{b_{x,y}\sigma_{x,y}}$$

$$= \sum_{\sigma,b} \prod_{x,y} K^{(\mathrm{delta})}_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y}b_{x,y+1}}$$

◆ index shift ($y + 1 \to y$) by δ

→ Tensor network constructed by $K^{(\mathrm{delta})}_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y}b_{x,y+1}} \equiv K_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y+1}} \times \delta_{b_{x,y}\sigma_{x,y}}$

◉ Schematic picture of the construction



$$K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}} \to K^{(\mathrm{delta})}_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y}b_{x,y+1}}$$

(1): Dots ↔ original indices, $\{\sigma_{x,y}, \sigma_{x+1,y}, \sigma_{x,y+1}\}$

(2): Draw arrows to connect all dots.

(3): Arrow (except σ) ↔ Index shift by δ.

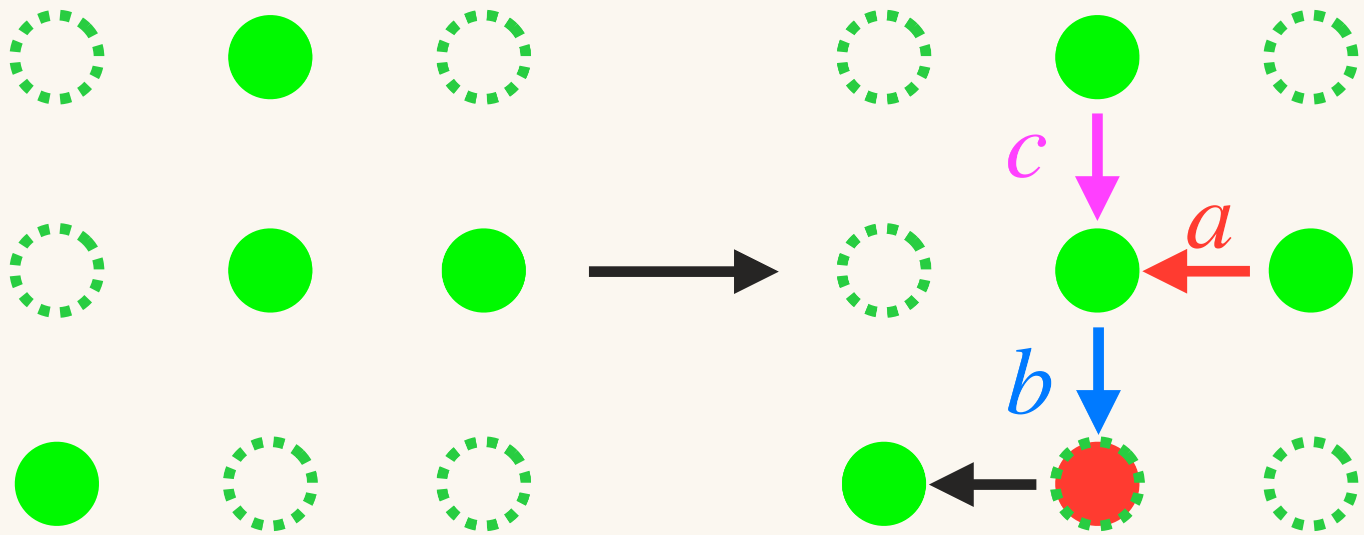
↓

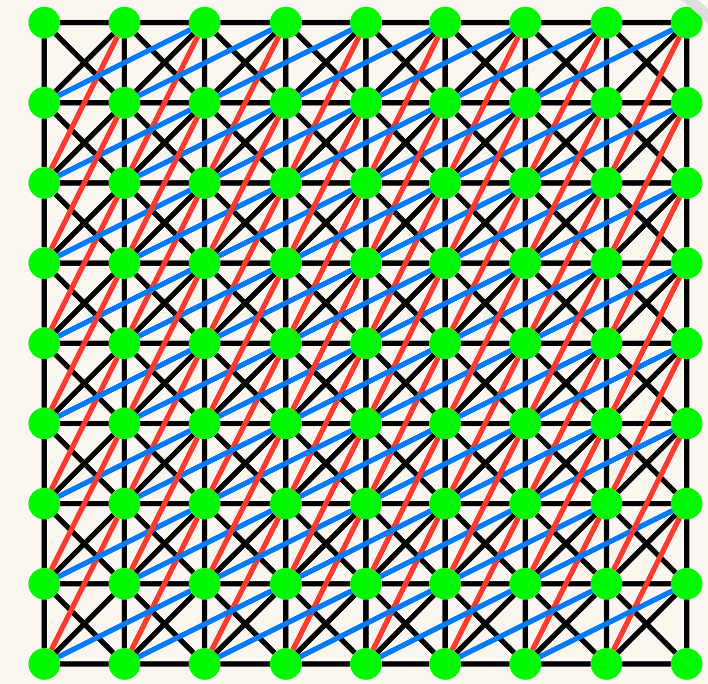
Arrows ↔ new indices, $\{\sigma, a\}$.

→ This method don't require any properties (equations).

(e.g. $\sigma^2 = 1$, $K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}} = e^{\sigma_{x,y}\sigma_{x+1,y}+\sigma_{x,y}\sigma_{x,y+1}}$)

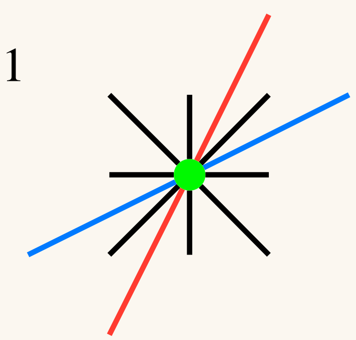
◉ Generalization: J1-J2+α





$$K_{\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+2,y+1}\sigma_{x+1,y+2}} \to K''_{[\sigma a]_{x,y}[\sigma a]_{x+1,y}[bc]_{x,y}[bc]_{x,y+1}}$$

$$e^{h(\sigma_{x,y}+\sigma_{x+1,y+1}+\sigma_{x+2,y+1}+\sigma_{x+1,y+2})}$$
$$e^{J_1(\sigma_{x+1,y+1}\sigma_{x+2,y+1}+\sigma_{x+1,y+1}\sigma_{x+1,y+2})}\, e^{J_2(\sigma_{x,y}\sigma_{x+1,y+1}+\sigma_{x+2,y+1}\sigma_{x+1,y+2})}\, e^{g_1\sigma_{x,y}\sigma_{x+2,y+1}+g_2\sigma_{x,y}\sigma_{x+1,y+2}}$$
$$e^{t_1\sigma_{x,y}\sigma_{x+2,y+1}\sigma_{x+1,y+2}+t_2\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+2,y+1}+t_3\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+1,y+2}+t_4\sigma_{x+1,y+1}\sigma_{x+2,y+1}\sigma_{x+1,y+2}}$$
$$e^{e_1\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+2,y+1}\sigma_{x+1,y+2}}$$



→ Added red dot is Steiner point.

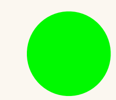
→ In general: Rectilinear Steiner tree problem (generalization of the traveling salesman problem).

→ K'' has only $4 \times 4 \times 4 \times 4$ indices.

◉ J1-J2+α: Frustrated system with 12 parameters.

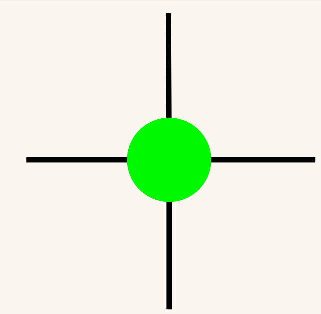
1-spin

$$e^{h(\sigma_{x,y}+\sigma_{x+1,y+1}+\sigma_{x+2,y+1}+\sigma_{x+1,y+2})}$$



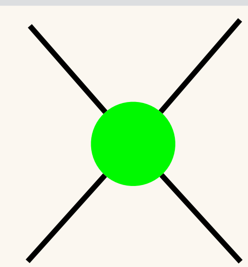
2-spin (J1)

$$e^{J_1^{(x)}\sigma_{x+1,y+1}\sigma_{x+2,y+1}+J_1^{(y)}\sigma_{x+1,y+1}\sigma_{x+1,y+2}}$$



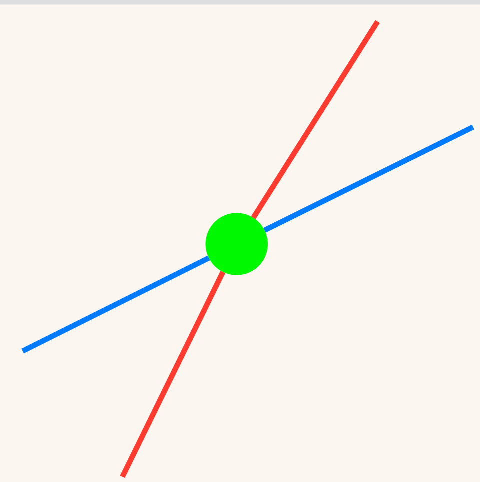
2-spin (J2)

$$e^{J_2^{(a)}\sigma_{x,y}\sigma_{x+1,y+1}+J_2^{(b)}\sigma_{x+2,y+1}\sigma_{x+1,y+2}}$$



2-spin (g1,g2)

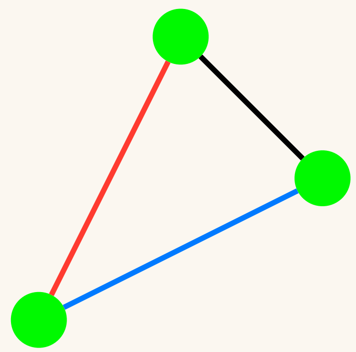
$$e^{g_1\sigma_{x,y}\sigma_{x+2,y+1}+g_2\sigma_{x,y}\sigma_{x+1,y+2}}$$



◉ J1-J2+α: Frustrated system with 12 parameters.

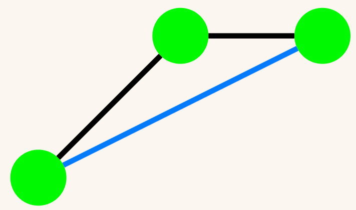
3-spin t1

$$e^{t_1 \sigma_{x,y} \sigma_{x+2,y+1} \sigma_{x+1,y+2}}$$



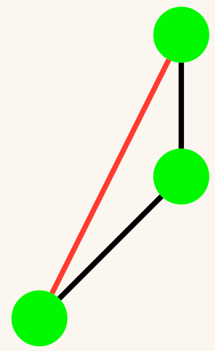
3-spin t2

$$e^{t_2 \sigma_{x,y} \sigma_{x+1,y+1} \sigma_{x+2,y+1}}$$



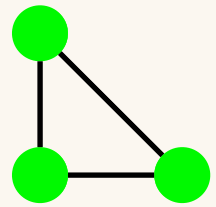
3-spin t3

$$e^{t_3 \sigma_{x,y} \sigma_{x+1,y+1} \sigma_{x+1,y+2}}$$



3-spin t4

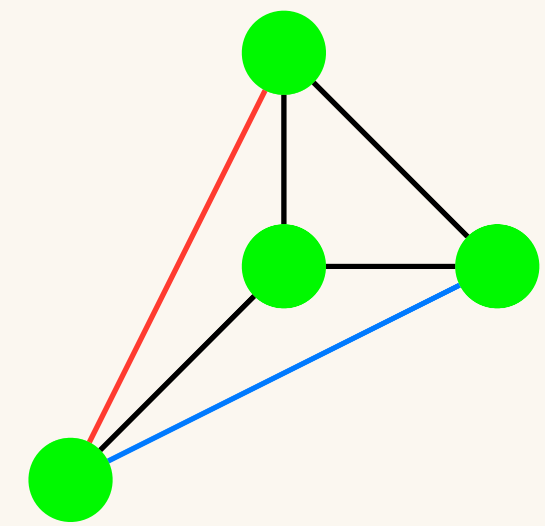
$$e^{t_4 \sigma_{x+1,y+1} \sigma_{x+2,y+1} \sigma_{x+1,y+2}}$$

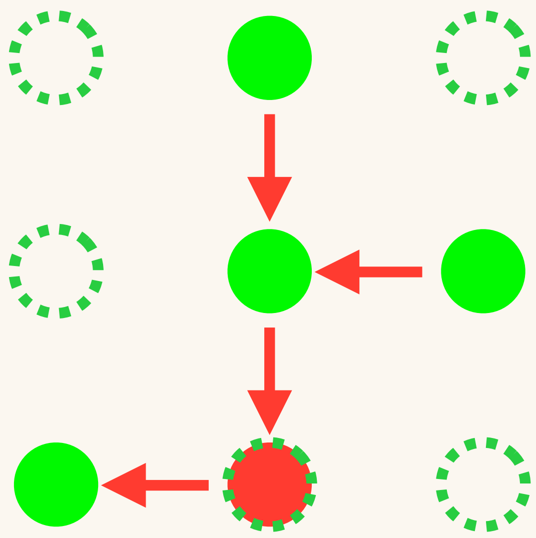


◉ J1-J2+α: Frustrated system with 12 parameters.

4-spin

$$e^{e_1 \sigma_{x,y} \sigma_{x+1,y+1} \sigma_{x+2,y+1} \sigma_{x+1,y+2}}$$





$$= K''_{[\sigma a]_{x,y}[\sigma a]_{x+1,y}[bc]_{x,y}[bc]_{x,y+1}}$$

◇ Num. of kind of interaction (parameters):

$$\left(\sum_{k=2}^{4} {}_4C_k \right) + 1 = 2^4 - 4 = 12$$

→ K'' has only $4 \times 4 \times 4 \times 4$ indices.

◉ Note: Steiner tree problem.

◇ Steiner tree problem:
Find shortest line segments (roads) between dots (towns) to connect every dots + we can freely add dots (town)

↓

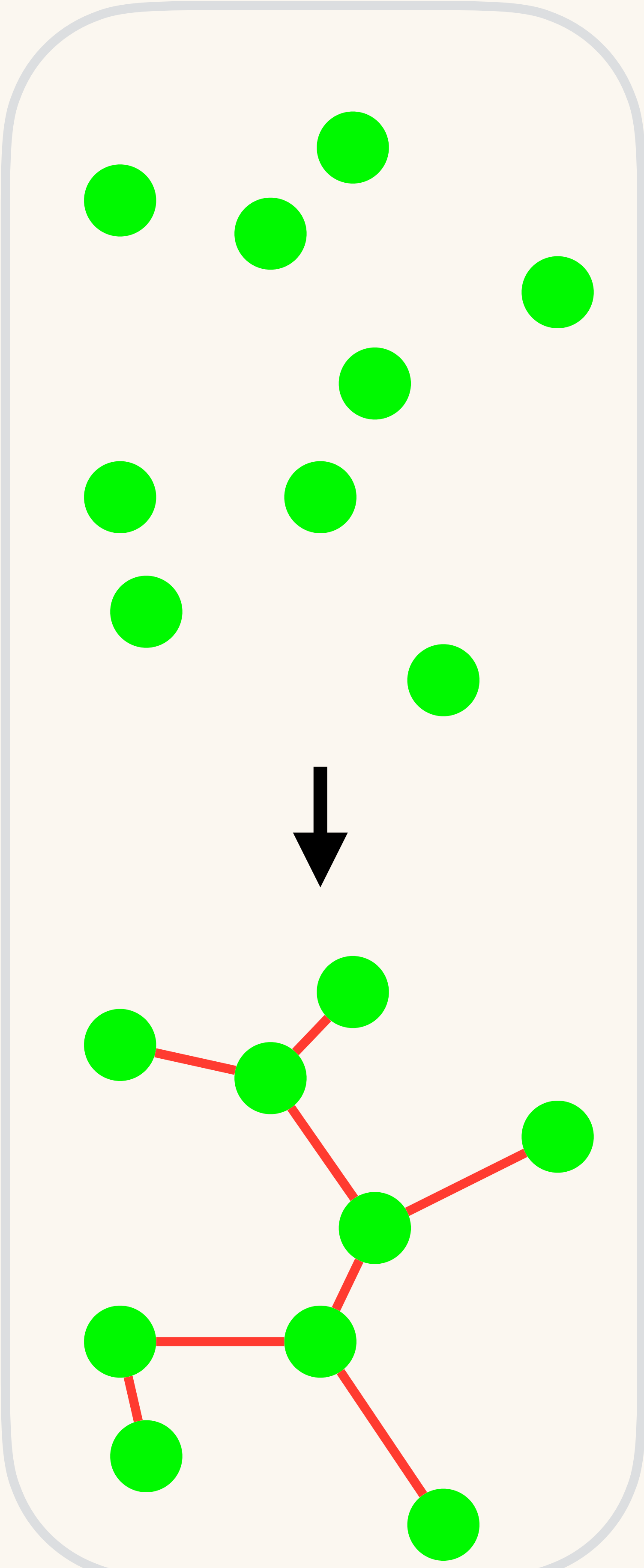
◇ Rectilinear Steiner tree problem: (NP-complete)
Steiner tree on the lattice. [M. Hanan, SIAM Appl. Math, 14, 2, p255, (1966)]
…dots on lattice points, line segments only on the links.

→ Index size of tensor = $D_{\mathrm{ini}}^{\text{(Road length)}}$

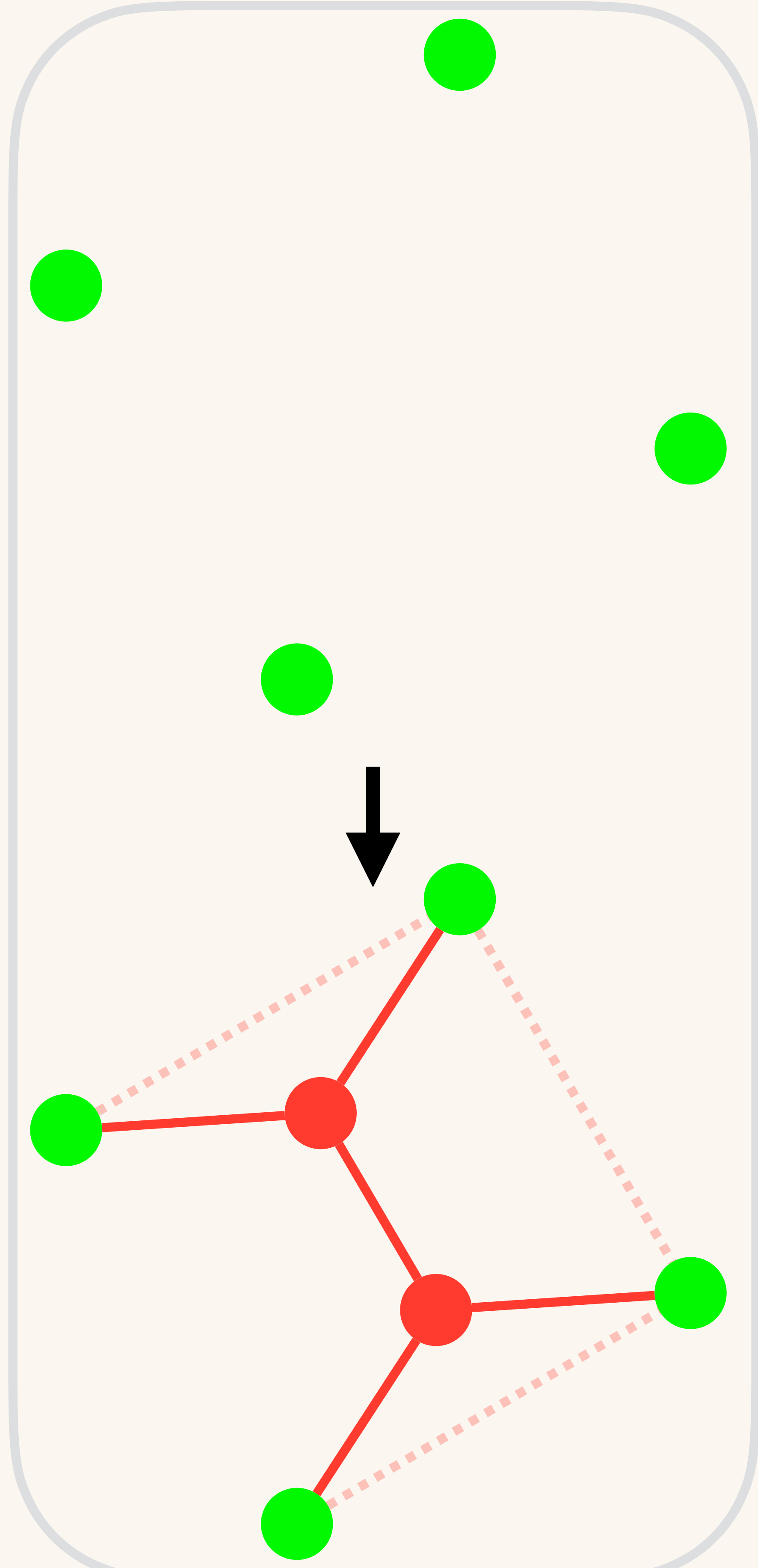
→ Long range interaction becomes harder.

→ Mild long ranges (e.g. plaquette, clover, etc.) are not hard.

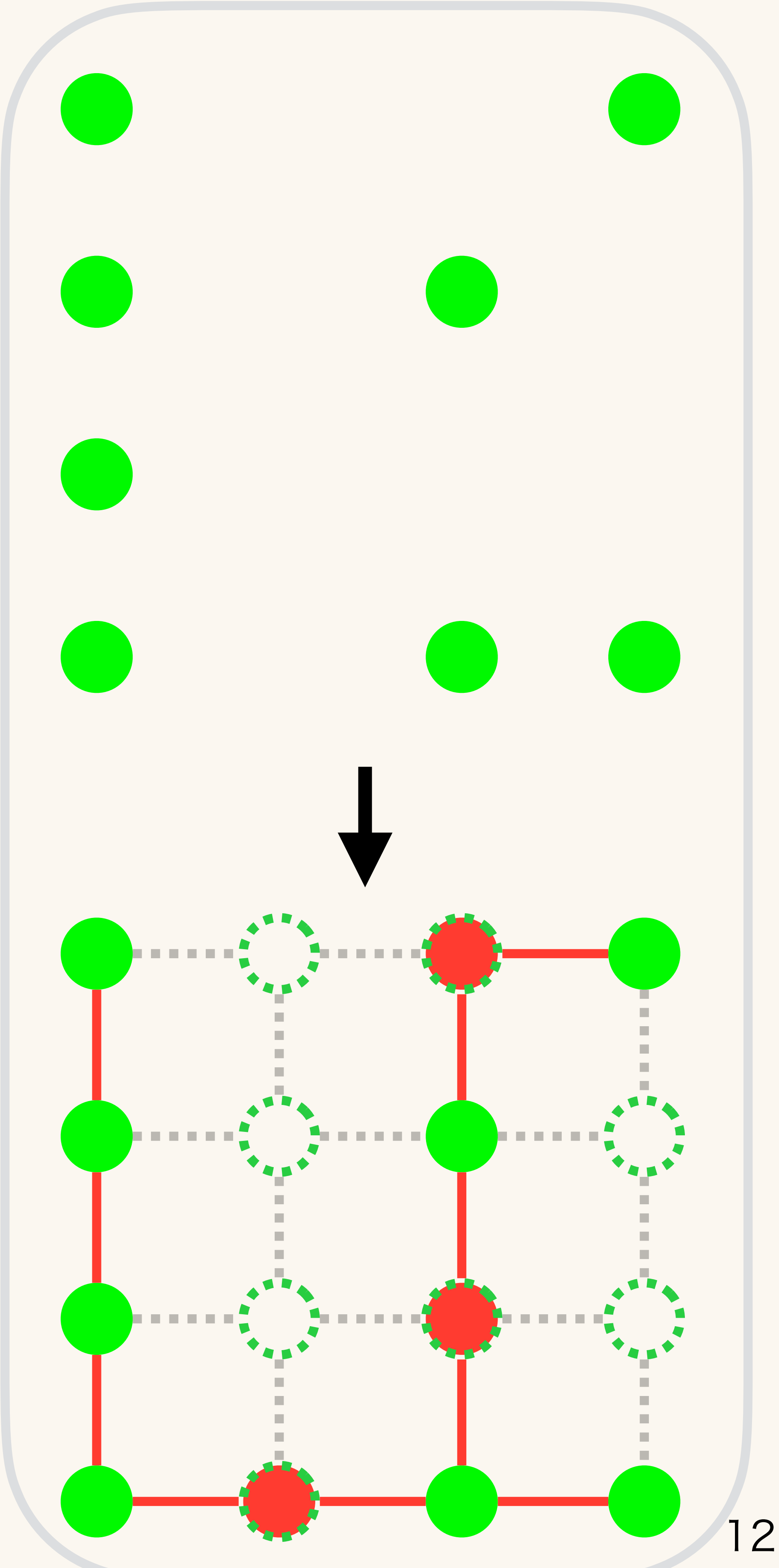
Minimum spanning



Steiner tree

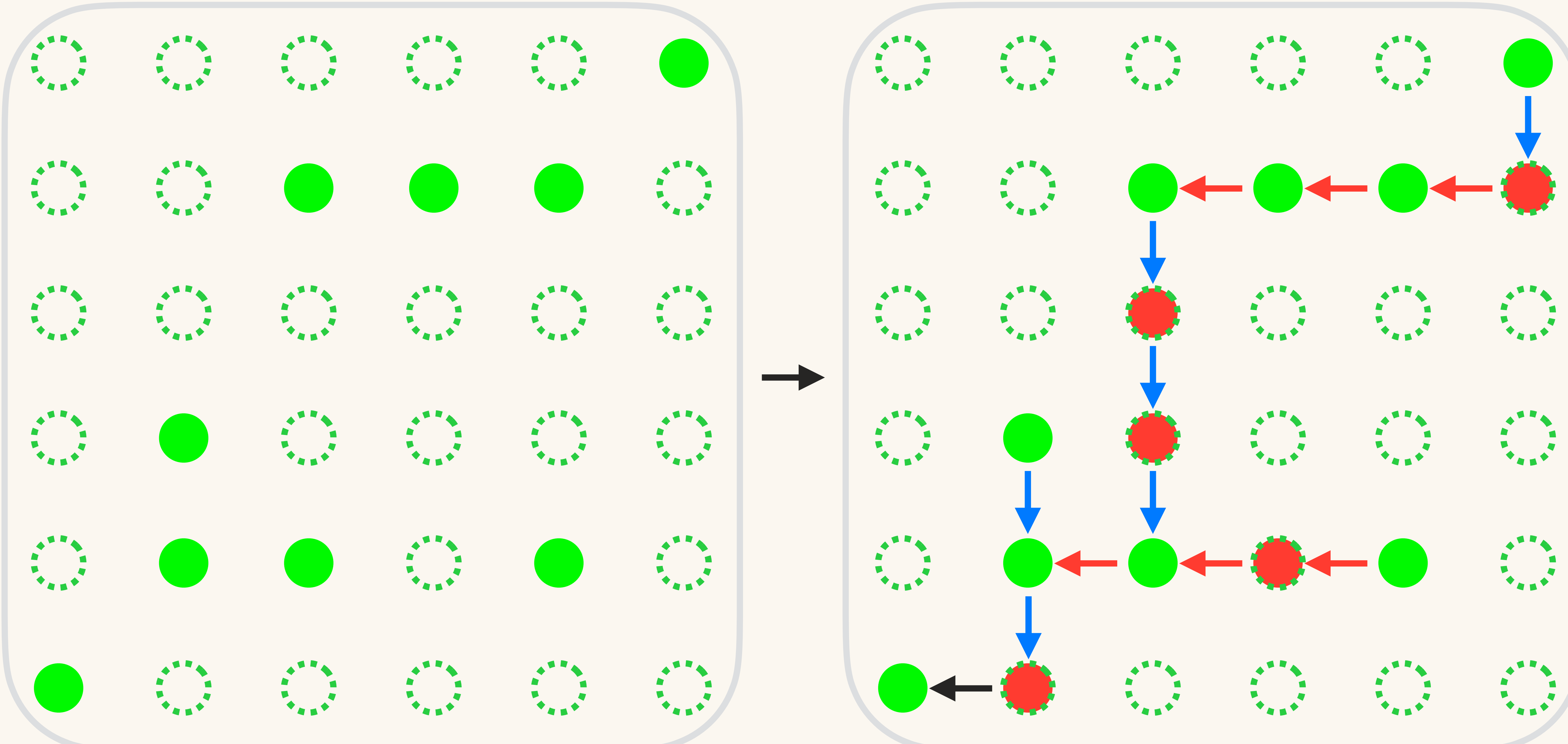


Rectilinear Steiner



◉ Note: (Rectilinear) Steiner tree problem

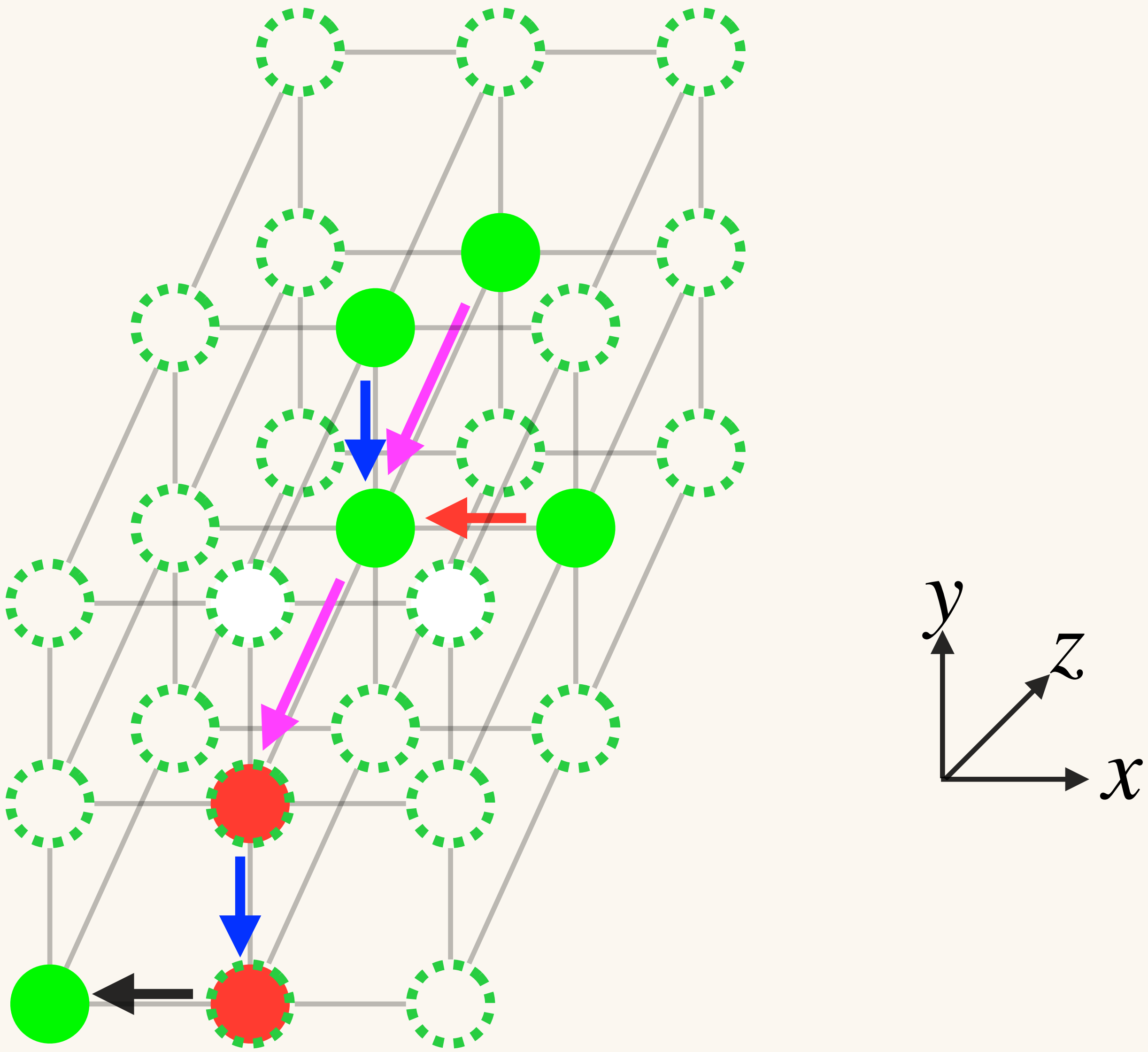
◇ Rectilinear Steiner tree problem: Find shortest line segment on the lattice between dots with additional dots.



→ Num. of interaction (parameters) order $\sim O(\sum {}_9C_k = 2^9)$

→K'' has $2^6 \times 2^6 \times 2^7 \times 2^7$ indices.

◉ (e.g.) J1-J2+α in 3dim



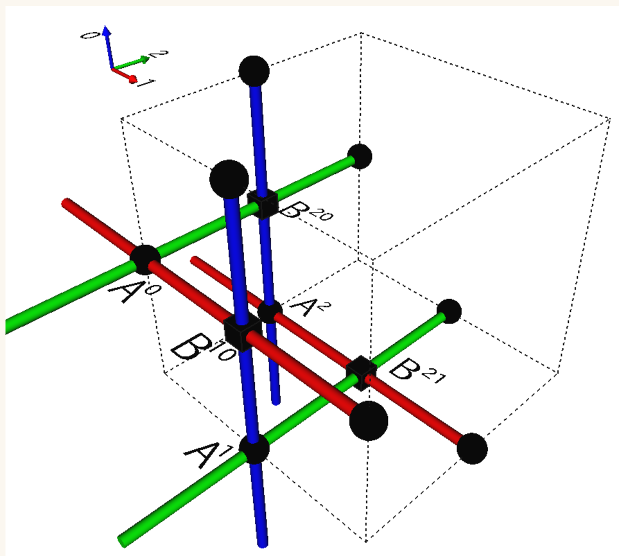
$$K^{(3d)}_{\sigma_{x,y,z}\sigma_{x+1,y+1,z+1}\sigma_{x+2,y+1,z+1}\sigma_{x+1,y+2,z+1}\sigma_{x+1,y+1,z+2}}$$

$$\to T^{(3d)}_{\sigma_{x,y,z}\sigma_{x+1,y,z}a_{x,y,z}a_{x+1,y,z}b_{x,y,z}b_{x,y+1,z}c_{x,y,z}c_{x,y+1,z}d_{x,y,z}d_{x,y,z+1}e_{x,y,z}e_{x,y,z+1}}$$

◉ Tensor network rep. of 3dim Z2 gauge theory

[Y.Liu et al. arXiv:1307.6543] [Y.Kuramashi, Y.Yoshimura, arXiv:1808.08025]

◇ Common method: (Taylor) expansion. and $\sigma^2 = 1$



$$Z = 2^{-3V} \sum_{\sigma} \prod_{n,\mu>\nu} e^{-\beta \sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu}}$$

$$e^{-\beta \sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu}} = \cosh\beta \sum_{p=0}^{1} (\tanh\beta)^p (\sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu})^p$$

$$A_{pqrs} = \mathrm{mod}(1 + p + q + r + s, 2)$$

$$B_{pqrs} = (\tanh\beta)^{(p+q+r+s)/4} \delta_{pq}\delta_{pr}\delta_{rs}$$

$$\downarrow$$

$$Z = \sum_{g,h,i,j,k,l} \prod_{n} T^{(\exp)}_{[gh]_{x,y,z}[gh]_{x+1,y,z}[ij]_{x,y,z}[ij]_{x,y+1,z}[kl]_{x,y,z}[kl]_{x,y,z+1}}$$

$$T^{(\exp)}_{[xX][x'X'][yY][y'Y'][zZ][z'Z']} = (\cosh\beta)^3 \sum_{a,b,c,d,e,f} A_{cyZe}A_{fzxb}A_{dYXa}B_{bx'y'c}B_{aX'Z'e}B_{fz'Y'd}$$

◇ In general: Any kind of expansion produces tensor network.

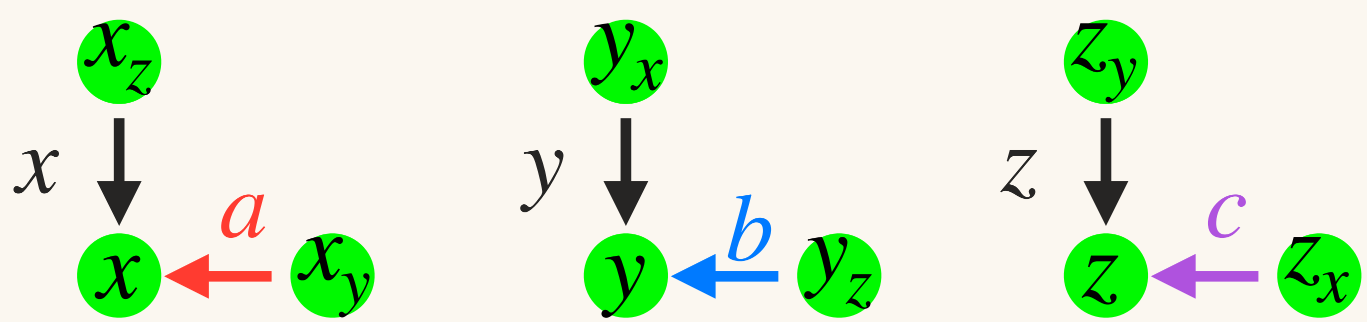
(Character for gauge theory, Orthogonal function, Taylor…)

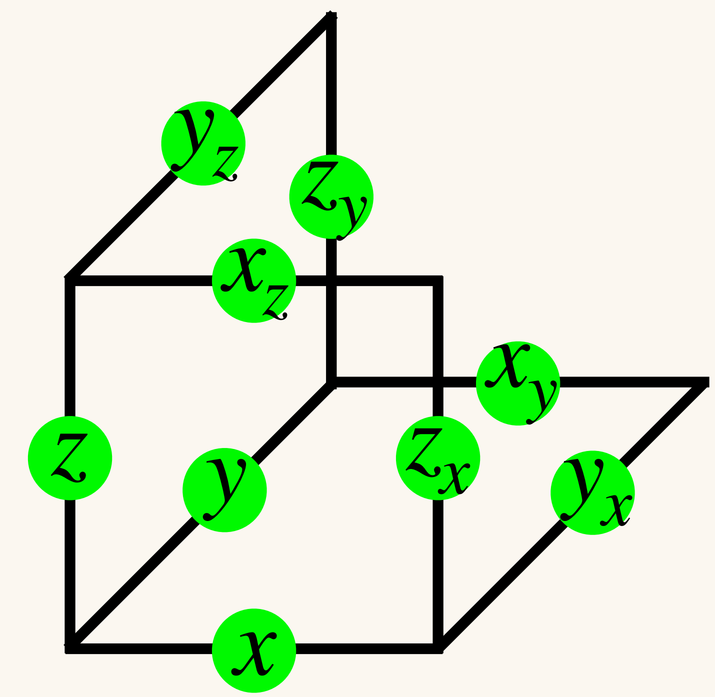
Tensor network rep. of 3dim Z2 gauge theory

◇ Our proposal: Index shift by delta matrix.

$$Z = 2^{-3V}\sum_{\sigma}\prod_{n,\mu>\nu} e^{-\beta\sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu}} = \sum_{\sigma}\prod_{n,\mu>\nu} e^{-\beta(xx_{\hat{y}}yy_{\hat{x}}+xx_{\hat{z}}zz_{\hat{x}}+yy_{\hat{z}}zz_{\hat{y}})/8}$$

→ x, y, z are independent with each other.



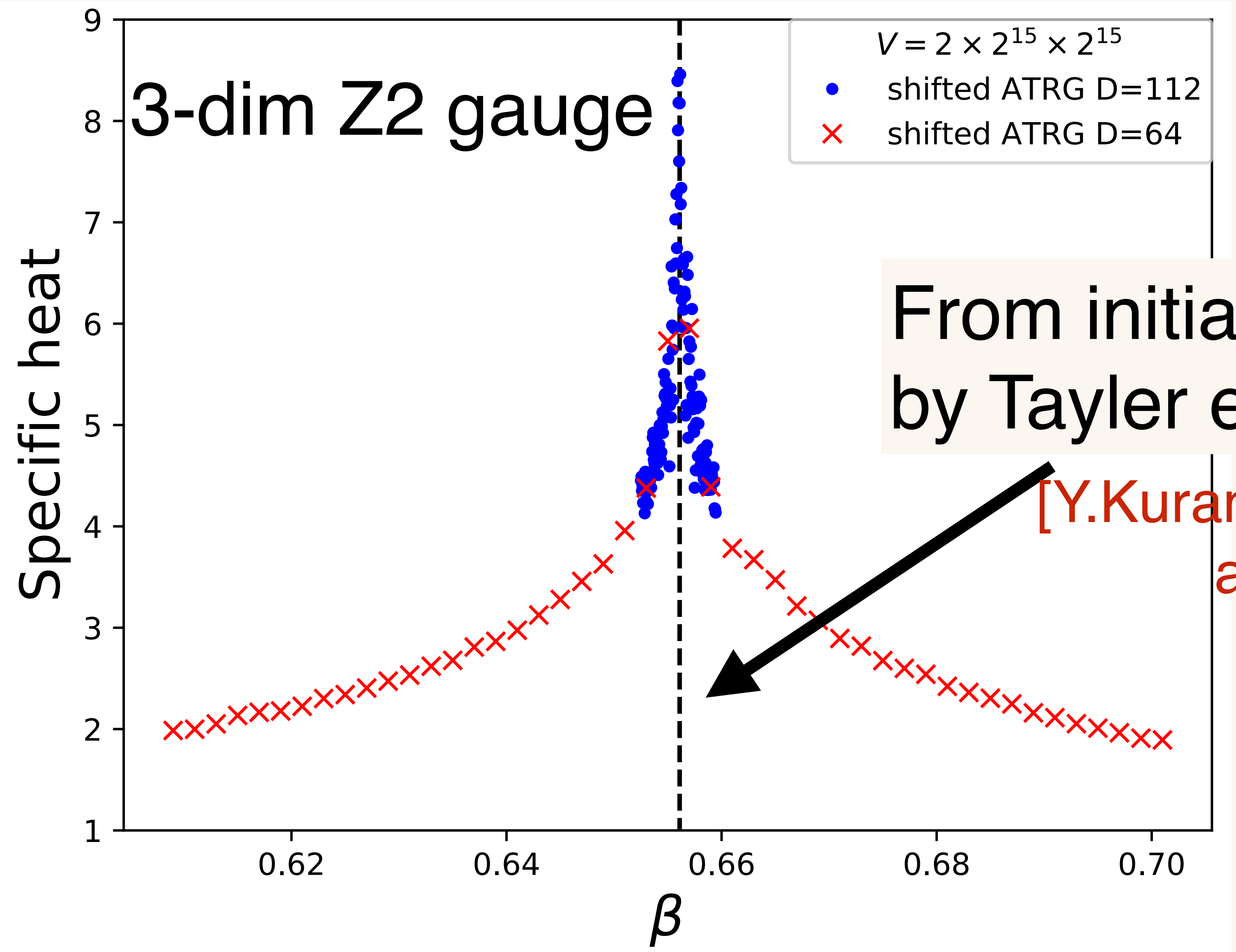


$$K_{xx_y x_z yy_z y_x zz_x z_y} \rightarrow K''_{[xb][xb]_z[yc][yc]_x[za][za]_y}$$

→ K'' has only $4 \times 4 \times 4 \times 4 \times 4 \times 4$ indices.

◉ Critical tempareture of 3dim Z2 gauge theory

◇ Numerical calc. by ATRG + impurity method



From initial tensor by Tayler expansion

[Y.Kuramashi et al. arXiv:1808.08025]

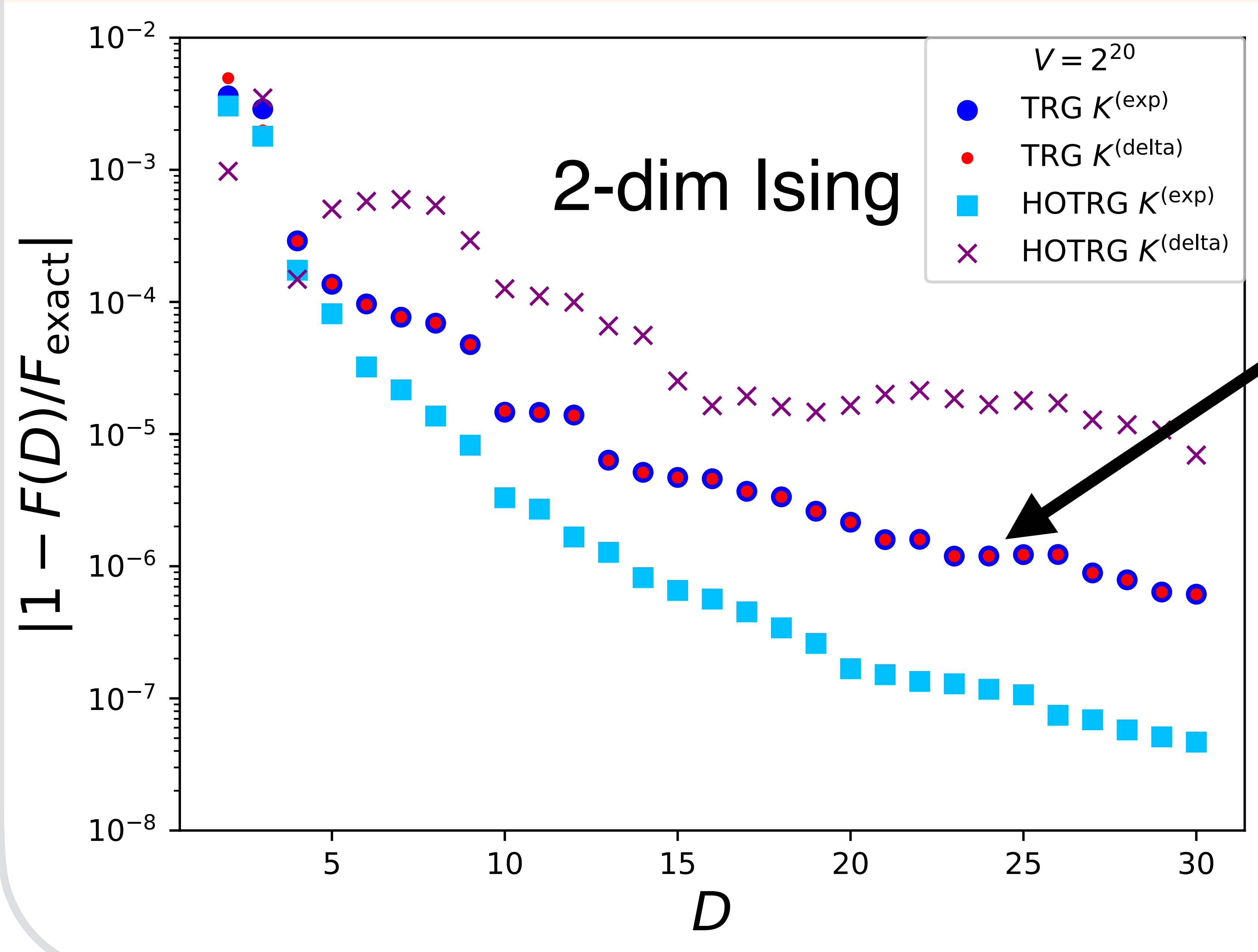
→ Our method produces correct result for critical temperature.

Initial tensor network dependence

Initial tensor dependence

◇ Now we can construct initial tensor by simple method.

→ We did not say our method produces best precision.



TRG has no dependence

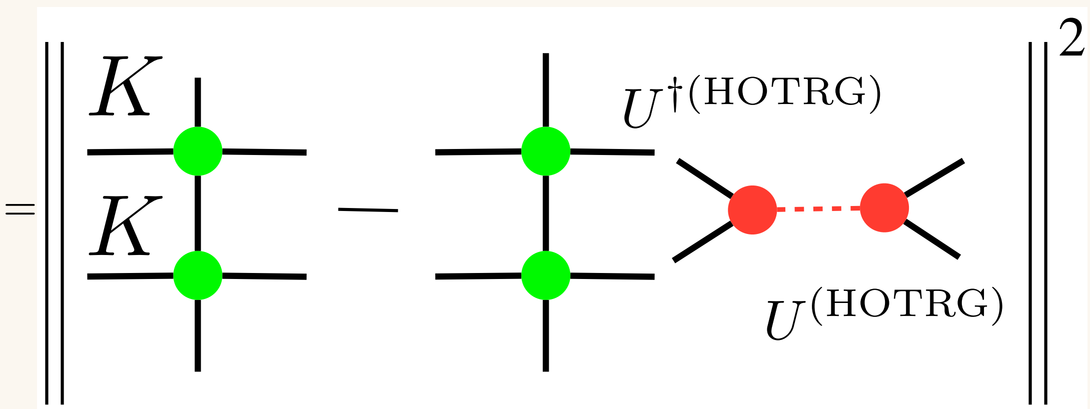
HOTRG has dependence

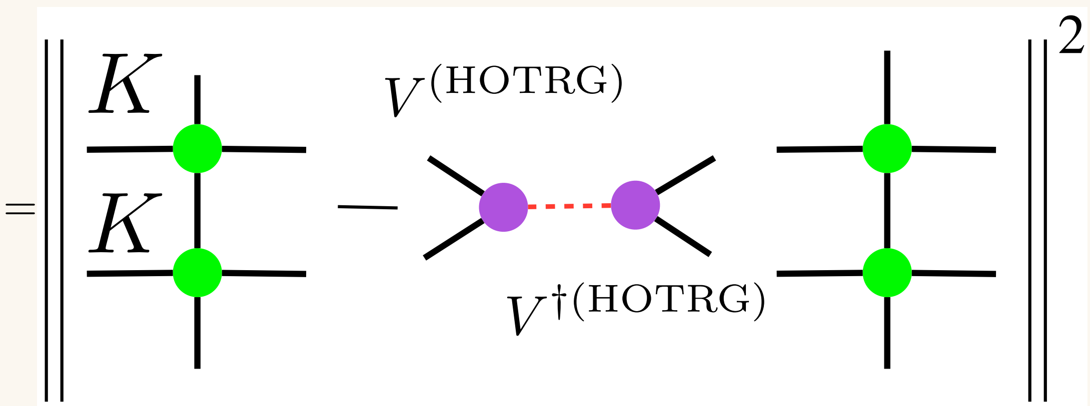
→ But we can eliminate this initial tensor dependence.

Boundary TRG

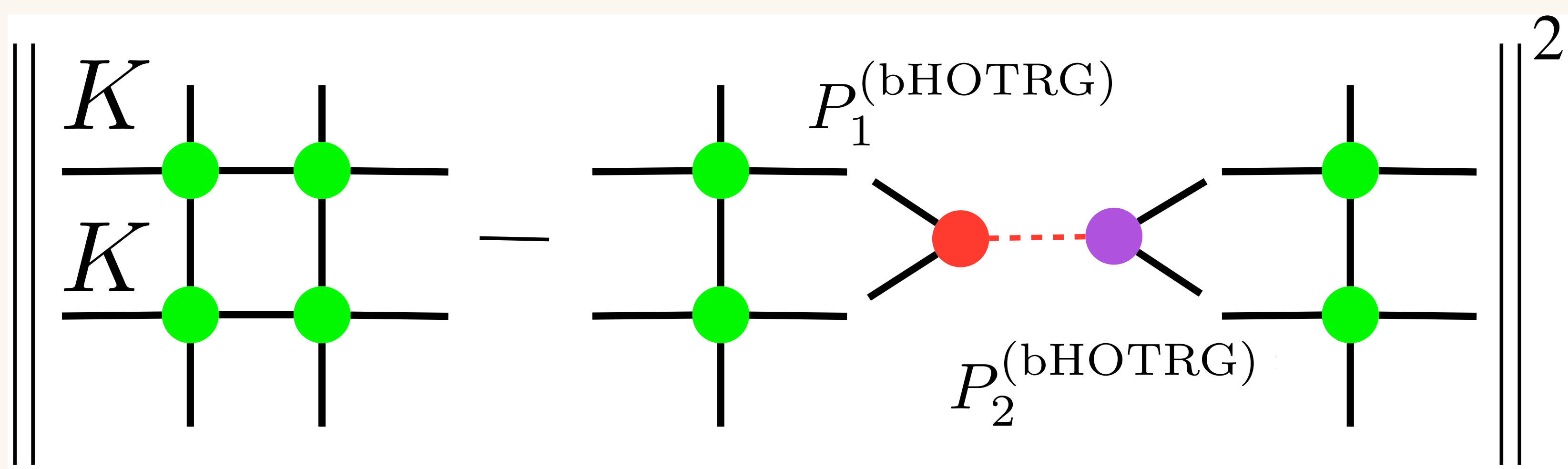
[S. Iino et al. arXiv:1905.02351]
[K. N., M.Schneider arXiv:2407.14226]

◇ Original HOTRG: choose U or V by minimizing the norm.

$C_U =$ 

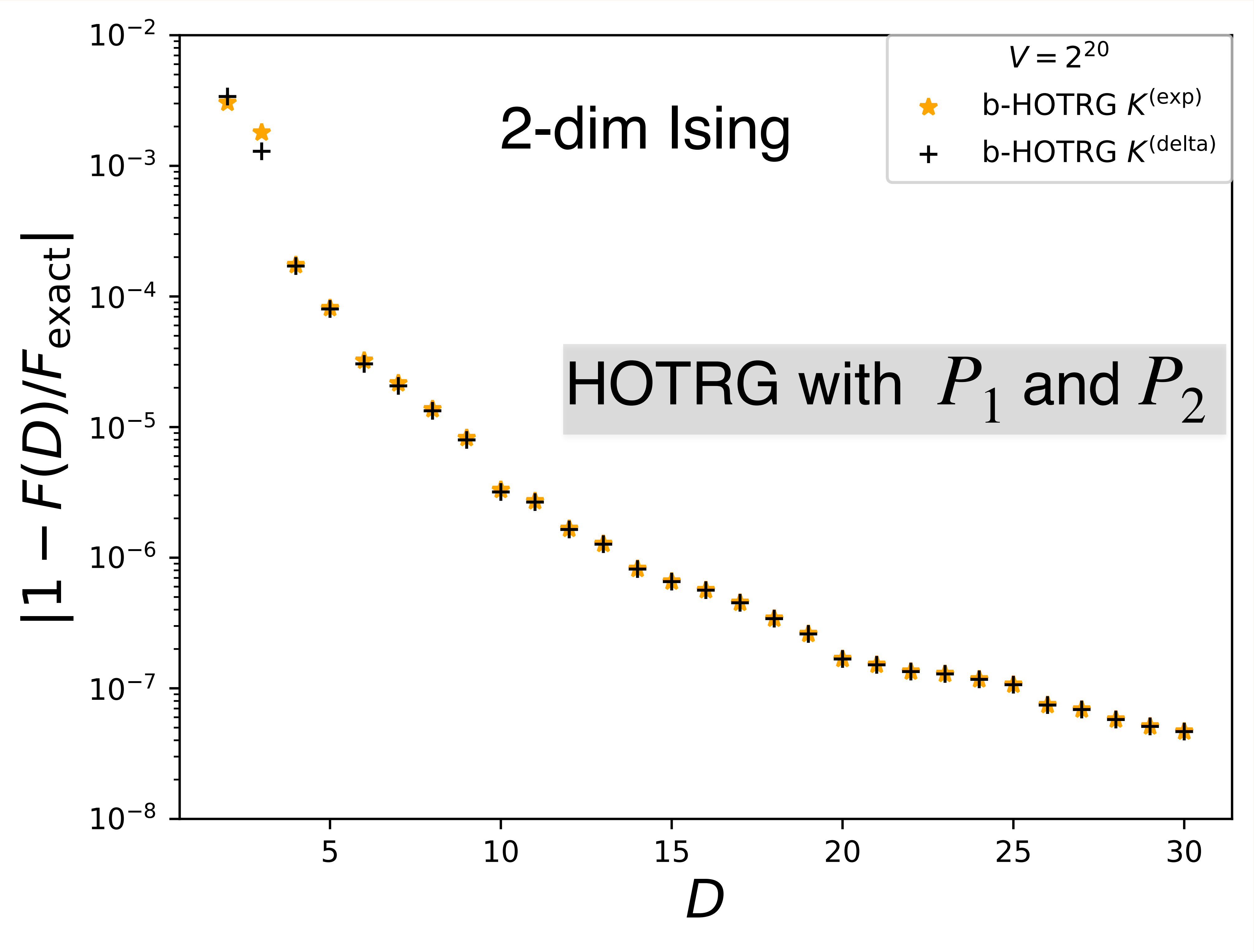
$C_V =$ 

◇ HOTRG with boundary TRG:
construct P_1, P_2 by U and V.

$C_{P_1,P_2} =$ 

→ Generalization from U or V to P_1 and P_2 can be done for any other TRG methods.

Boundary HOTRG



→ HOTRG with P_1 and P_2 is initial tensor independent.

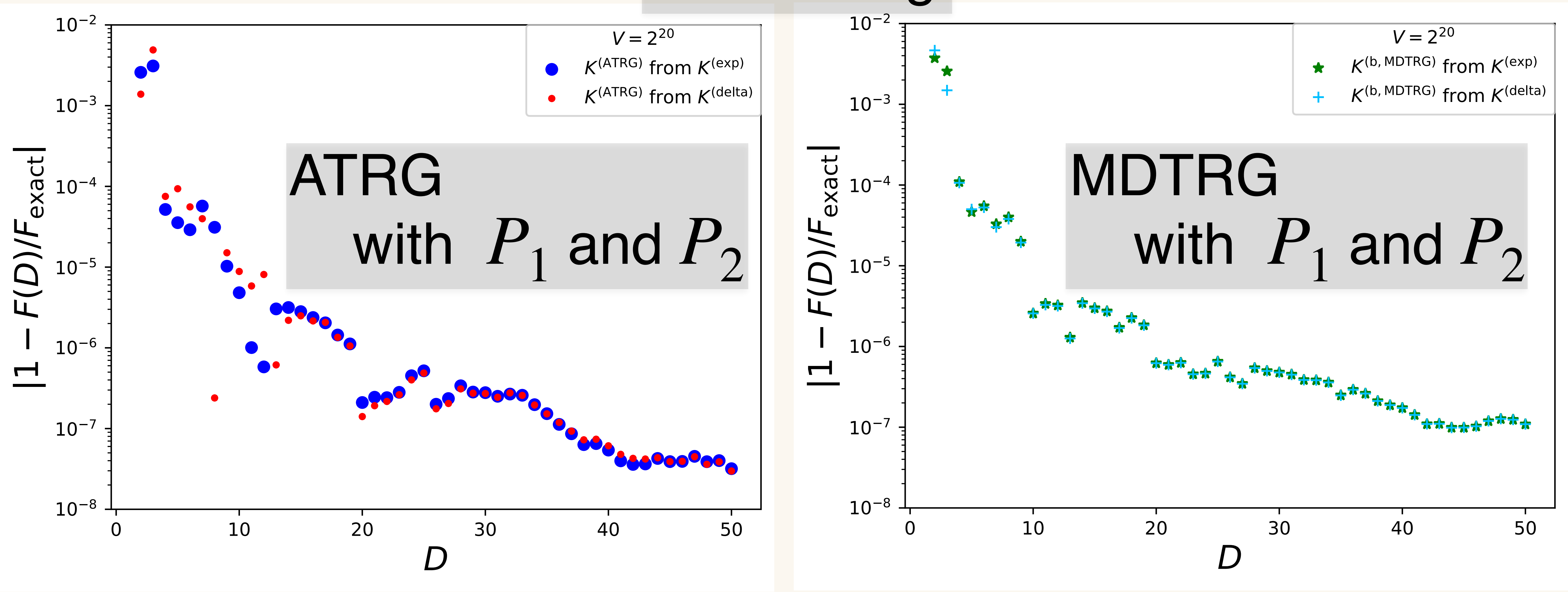
◉ Boundary TRG for ATRG and MDTRG

[D. Adachi et al. arXiv:1906.02007]
[D. Kadoh, K.N. arXiv:1912.02414] [K.N. arXiv:2307.14191]

- ◇ Many method (ATRG, TriadTRG, MDTRG) can use isometry (U or V) or P_1 and P_2.

[K. N., M.Schneider arXiv:2407.14226]

2-dim Ising



→ TRG methods with P_1 and P_2 are initial tensor independent.

→ Our construction can also produce compatible result.

◉ Summary

[main]

- ◇ We propose a general method to construct the initial tensor network from Boltzmann factor representation.

- ◇ We can eliminate initial tensor dependence by using boundary TRG method for isometry.

[details]

- ◇ This relates to the rectilinear Steiner tree problem (relates optimization prob. such as traveling salesman)

- ◇ We test our construction in the 2-dim Ising model and 3-dim Z2 gauge theory, and reproduce results.

→ Our method could be a simple, good choice for first study.