TN representation of non-abelian gauge theory coupled to reduced staggered fermions

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Tensor Network 2024 (TN24)
Shiinoki Cultural Complex + Online

Complex action problems (sign problems) in lattice QFT

- In expectation value of a physical quantity $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \mathcal{O}[\varphi] \, e^{-S[\varphi]},$$

  - φ, S: field, action
  - Z: partition function

  the Boltzmann factor is not necessarily positive definite;

$$\frac{1}{Z} e^{-S[\varphi]} \in \mathbb{C}.$$

- Supersymmetric systems, chiral gauge theories, finite density QCD are difficult in probabilistic treatments.
- Strong motivation to apply tensor network approaches
- Other motivations: go well with quantum computing, real time evolution, etc...

TN approaches to non-abelian gauge theories

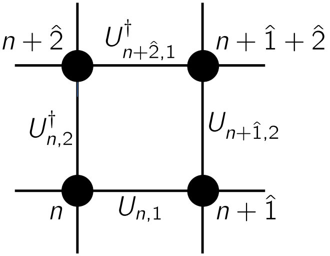
- The numerical complexity of tensor methods strongly depends on the number of physical d.o.f.
- Non-abelian gauge theories typically have huge internal d.o.f.
- Coupling to matter fields makes the situation more serious.
- Works by others:
  - Character expansion [Liu et al., 2013; Hirasawa et al., 2021; Bazavov et al., 2019; Asaduzzaman et al., 2020]
  - Probabilistic sampling [Fukuma et al., 2021]
  - Trial (variational) actions [Kuwahara and Tsuchiya, 2022]
  - Using only representation indices [Yosprakob, 2024]
  - 2D QCD in strong coupling limit [Bloch and Lohmayer, 2023]
- In this work, we see how simple discretization + truncation of gauge d.o.f. works and if coupling to fermions is acceptable.

Pure $SU(2)$ LGT

- The lattice action is given by

$$S = -\frac{\beta}{2} \sum_n \text{tr}\left[U_{n,1} U_{n+\hat{1},2} U^\dagger_{n+\hat{2},1} U^\dagger_{n,2} \right].$$

  - β: inverse coupling, $U_{n,\mu} = \exp\left\{ igA^i_{n,\mu} T^i \right\}$: link variable, T: generator of $SU(2)$



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- The partition function is

$$\begin{aligned} Z &= \int \mathcal{D}U e^{-S} \\ &= \int \mathcal{D}U \prod_{n} e^{(\beta/2)\, \mathrm{tr}\left[U_{n,1} U_{n+\hat{1},2} U^{\dagger}_{n+\hat{2},1} U^{\dagger}_{n,2} \right]}. \end{aligned}$$

  - $\mathcal{D}U = \prod_n \mathrm{d}U_{n,1} \mathrm{d}U_{n,2}$: $SU(2)$ Haar measure

Parameterization for $SU(2)$ elements

- To consider the integral of the gauge variables, we use the following parameterization of the gauge elements

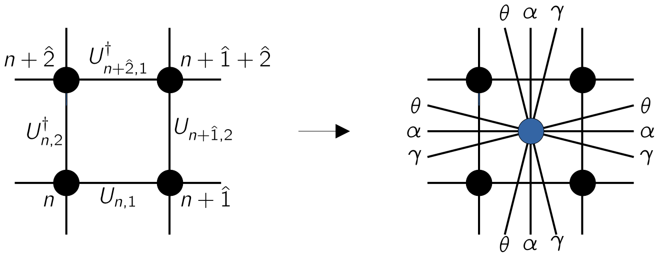
$$U_{n,\mu}(\theta,\alpha,\gamma) = \begin{pmatrix} \cos\theta_{n,\mu} e^{i\alpha_{n,\mu}} & \sin\theta_{n,\mu} e^{i\gamma_{n,\mu}} \\ -\sin\theta_{n,\mu} e^{-i\gamma_{n,\mu}} & \cos\theta_{n,\mu} e^{-i\alpha_{n,\mu}} \end{pmatrix}.$$

- Under this parameterization, the integral for gauge variables becomes

$$\int \mathcal{D}U = \int \prod_{n,\mu} \mathrm{d}U_{n,\mu} = \prod_{n,\mu} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta_{n,\mu} \int_{-\pi}^{\pi} \mathrm{d}\alpha_{n,\mu} \int_{-\pi}^{\pi} \mathrm{d}\gamma_{n,\mu} \frac{\sin\theta_{n,\mu}\cos\theta_{n,\mu}}{2\pi^2}.$$

How to extract discrete d.o.f.?

- Now each link has 3 variables (angles).
- Then each plaquette can be regarded as (3×4)-rank continuously indexed tensor.
- Next step is extracting discrete d.o.f.
  - Taylor expansion (does not so rapidly converge)
  - character expansion (not clear when coupled to fermions)
  - Gaussian quadrature rule



Gaussian quadrature rule

- Approximate integral as weighted summation (replace the integration variable by discrete nodes)
- In Gauss–Legendre quad., roots of Legendre polynomial are used as nodes.

$$\int_a^b \mathrm{d}\phi g(\phi) \approx \frac{b-a}{2} \sum_{i=1}^{K} w_i g\left(\frac{b-a}{2} x_i + \frac{a+b}{2}\right)$$

  - $g(\phi)$: (well-behaved) arbitrary integrand [1]
  - K: Order of Legendre polynomial
  - x_i: i-th root of Legendre polynomial
  - w_i: i-th weight of summation
- Using this we can replace the field variable by a discrete one at each site or link: $\{\phi\} \to \{x\}$.

[1] If g is a polynomial of order $2K-1$ or less, GL quad. is exact.

Discretization of plaquette

- By applying GL quadrature for every angle, plaquette tensor can be discretized:

$$
\begin{aligned}
P_{(ijk)(lmn)(opq)(rst)} &= \prod_{a,b,c,d=1}^{2} e^{(\beta/2)U^{bc}U^{dc*}U^{ad*}U^{ab}} \\
&= \prod_{a,b,c,d=1}^{2} \exp\left\{ \frac{\beta}{2} U\left(\frac{\pi}{4}x_i + \frac{\pi}{4}, \pi x_j, \pi x_k\right)_{bc} U\left(\frac{\pi}{4}x_l + \frac{\pi}{4}, \pi x_m, \pi x_n\right)^*_{dc} \right. \\
&\qquad \left. \cdot U\left(\frac{\pi}{4}x_o + \frac{\pi}{4}, \pi x_p, \pi x_q\right)^*_{ad} U\left(\frac{\pi}{4}x_r + \frac{\pi}{4}, \pi x_s, \pi x_t\right)_{ab} \right\},
\end{aligned}
$$

- Now P can be regarded as a 12-rank tensor, and the number of elements is K^{12}.

$\Rightarrow$ For $K > 10$, the size of P reaches $\mathcal{O}(\mathrm{TB})$.

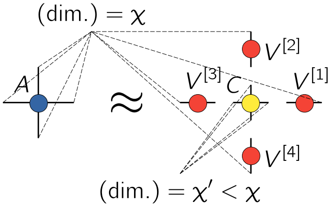
- *Cf.* Precise determination for criticality of real ϕ^4 theory was done with $K = 1024$ [Kadoh et al., 2019].

Higher-order orthogonal iteration (HOOI) [De Lathauwer et al., 2000]

0 Input: an N-rank tensor A whose bond dimension is χ.
Output: a core tensor C whose bond dimension is $\chi' < \chi$, and a set of unitary matrices V whose dimension is $\chi' \times \chi$, so that the tensor

$$X_{I_1 I_2 \cdots I_N} = \sum_{i_1, i_2, \ldots, i_N = 1}^{\chi'} C_{i_1 i_2 \cdots i_N} V^{[1]}_{i_1 I_1} V^{[2]}_{i_2 I_2} \cdots V^{[N]}_{i_N I_N}$$

approximates A well. For the simplicity, here we assume that the length of each direction is the same for each A and C.



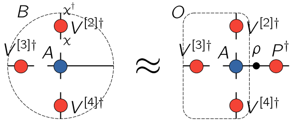
Higher-order orthogonal iteration (HOOI) [De Lathauwer et al., 2000]

1. Initialize Vs as randomly generated unitary matrices.
2. For j-th leg each,
   - Apply $V^{[\tilde{j}]\dagger}$s to A for $\tilde{j} \neq j$:

$$B_{i_1 i_2 \cdots l_j \cdots i_N} = \sum_{l_1, l_2, \ldots, l_{j-1}, l_{j+1}, \ldots, l_N = 1}^{\chi} A_{l_1 l_2 \cdots l_N} V^{[1]\dagger}_{l_1 i_1} V^{[2]\dagger}_{l_2 i_2} \cdots V^{[j-1]\dagger}_{l_{j-1} i_{j-1}} V^{[j+1]\dagger}_{l_{j+1} i_{j+1}} \cdots V^{[N]\dagger}_{l_N i_N}.$$

   - Take a truncated singular value decomposition (SVD) along the j-th leg of B:

$$B_{i_1 i_2 \cdots l_j \cdots i_N} \approx \sum_{k=1}^{\chi'} O_{i_1 i_2 \cdots k \cdots i_N} \rho_k P^{\dagger}_{k l_j}.$$



   - Update $V^{[j]}$ by $P^{\dagger}$.

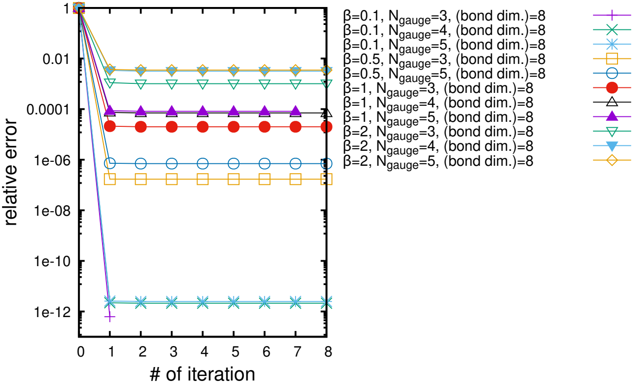
Higher-order orthogonal iteration (HOOI) [De Lathauwer et al., 2000]

3. Update C as

$$C_{i_1 i_2 \cdots i_N} = \sum_{I_1, I_2, \ldots, I_N = 1}^{\chi} A_{I_1 I_2 \cdots I_N} V^{[1]\dagger}_{I_1 i_1} V^{[2]\dagger}_{I_2 i_2} \cdots V^{[N]\dagger}_{I_N i_N}.$$

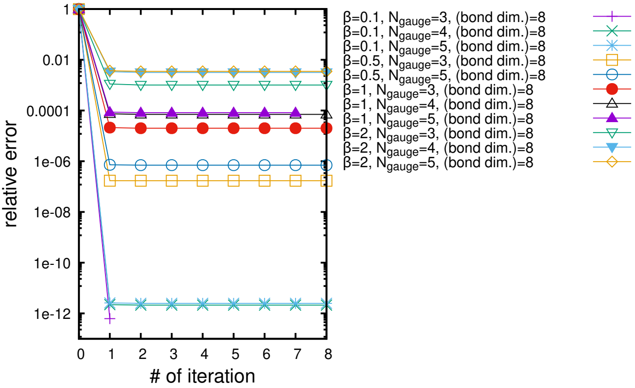
4. Iterate until the error $|A - X|_{\mathrm{F}} / |A|_{\mathrm{F}}$ converges, where $|\cdot|_{\mathrm{F}}$ denotes the Frobenius norm.

HOOI for plaquette tensor



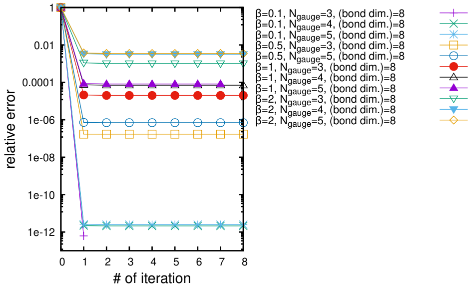
$$\frac{|P - CV^{[1]}V^{[2]}V^{[3]}V^{[4]}|_F}{|P|_F}.$$

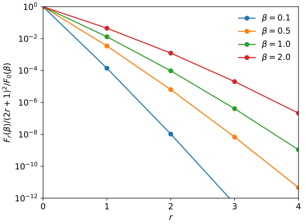
HOOI for plaquette tensor



- HOOI reduces the number of elements from N_{gauge}^{12} to 8^4.
- Strong coupling region $\beta < 0.5$ is quite accurate.

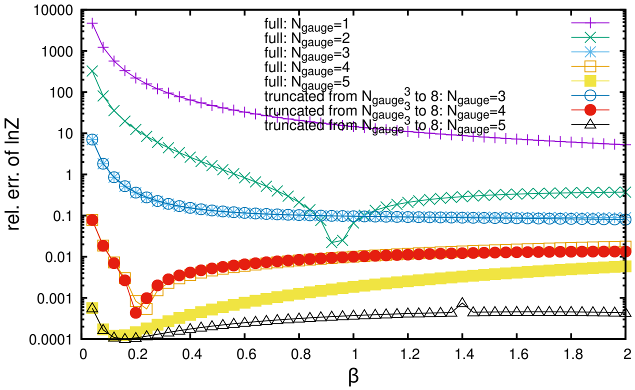
HOOI for plaquette tensor





- *Cf.* Decay of weight in the model expanded by character.
- $F_r(\beta) = I_{2r}(\beta) - I_{2r+2}(\beta)$.
- Qualitatively common; large β leads to milder decay.
- Note that (right) is weight in Z itself, while (left) shows error of plaquette.

Free energy of pure $SU(2)$ theory



- Relative error of $\ln Z$ on $L = 4$ lattice.
- $N_{\text{gauge}} = 1$ corresponds to an 1D PCA.
- Effect of HOOI is surprisingly mild.

Reduced staggered fermion formulation [Doel and Smit, 1983]

- The full staggered fermion action is

$$S_{\mathrm{F}}[U] = \sum_n \sum_{\mu=1}^{2} \frac{\eta_{n,\mu}}{2} \left(\bar{\psi}_n U_{n,\mu} \psi_{n+\hat{\mu}} - \bar{\psi}_{n+\hat{\mu}} U^{\dagger}_{n,\mu} \psi_n \right),$$

where $\eta_{n,\mu} = (-1)^{\sum_{\nu<\mu} n_\nu}$.

- By transforming $\psi_n \to (1-\epsilon_n)\psi_n/2$ and $\bar{\psi}_n \to (1+\epsilon_n)\bar{\psi}_n/2$ with the parity factor $\epsilon_n = (-1)^{n_1+n_2}$, ψ_n and $\bar{\psi}_n$ live only on even and odd site, respectively, or vise versa.
- Thus one can relabel $\bar{\psi}_n$ as ψ_n^{T}, and the reduced staggered action is

$$S_{\mathrm{F}}[U] = \sum_n \sum_{\mu=1}^{2} \frac{\eta_{n,\mu}}{2} \psi_n^{\mathrm{T}} \mathcal{U}_{n,\mu} \psi_{n+\hat{\mu}},$$

where the "projected " link variable is defined by $\mathcal{U}_{n,\mu} = (1+\epsilon_n) U_{n,\mu}/2 + (1-\epsilon_n) U^*_{n,\mu}/2$.

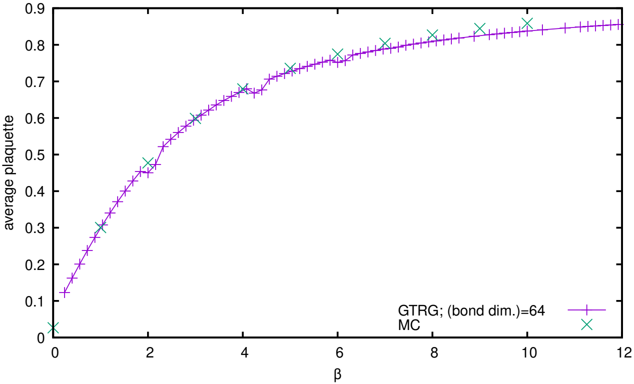
Total partition function

- The total partition function takes a form like

$$Z = \sum_{\text{(all indices)}} T_{\mathrm{F}} T_{\mathrm{G}}.$$

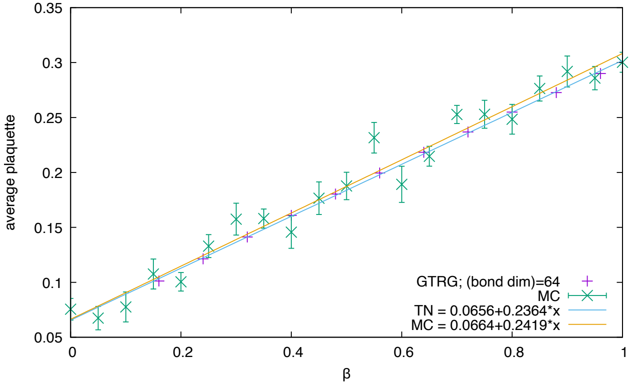
- Note that the fermion part T_F and the gauge part T_G share indices with each other.
- The bond dimension of T_{F} is 16 (with the reduced staggered formulation), and that of T_{G} is 8 (with HOOI), so the total tensor has $(16 \times 8)^4$ elements in our actual calculation.
- (This is $\mathcal{O}(\mathrm{GB})$ with double precision.)

Average plaquette on $L = 32$



- $SU(2)$ LGT coupled to fermions.
- Comparison to RHMC. (Free from sign problem for $SU(2)$.)
- Roughly agree with each other, but small deviation at large β (weak coupling).

Closer look at strong coupling region



- Analytical form is (ave. plaq.) $= (1/2)^4 + \beta/4$. (Intercept term arises from the fermion part.)
- TN result is much stable.

Summary

- Economical construction of TN for $SU(2)$ LGT w/ fermions.
- Drastic reduction of d.o.f., but reasonably accurate.

In future

- Tackle to nontrivial models w/ sign problems, *e.g.* adding topological term.
- Higher dimensions.
- Check how tolerable it is for general $SU(N)$.
- Comparison to the other approaches:
  - Character expansion [Liu et al., 2013; Hirasawa et al., 2021; Bazavov et al., 2019; Asaduzzaman et al., 2020]
  - Probabilistic sampling [Fukuma et al., 2021]
  - Trial (variational) actions [Kuwahara and Tsuchiya, 2022]
  - Using only representation indices [Yosprakob, 2024]
  - 2D QCD in strong coupling limit [Bloch and Lohmayer, 2023]
- Key point would be coupling to matters.

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