Quantum algorithms for simulating the Schwinger model

Based on Quantum 8, 1474 (2024), arXiv:2311.17388

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Kazuki Sakamoto

Graduate School of Engineering Science, Osaka University

Today's Content

1. Quantum algorithms for simulating quantum systems
   - Why do we use quantum computers?
   - Trotter formula (Conventional method)
   - Quantum singular value transformation (Recent method)

2. Application to particle physics
   - Our recent work:
     efficient quantum algorithm for simulating the Schwinger model

Quantum algorithms for simulating quantum systems

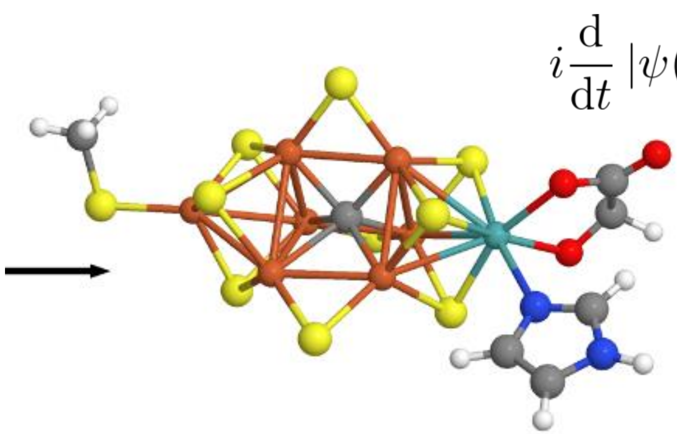
Introduction to quantum computers

Computers which utilize the properties of quantum systems

➢ Quantum computers are expected to obtain (**exponential) quantum speed ups** over classical computers for some problems.

- **Quantum simulation** S. Lloyd, Science, (1996).

$$i\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$



M. Reiher, et al. PNAS, (2017).

- Condensed matter physics
- Quantum chemistry
- High energy physics

- **Linear algebraic problems**

A. W. Harrow, Phys. Rev. Lett., (2009).

$$A\left|x\right\rangle = \left|b\right\rangle$$

- Factoring
- Fluid dynamics simulation
- Machine learning
- Data analysis

Introduction to quantum computers

Components of a quantum computer

Qubit

- 2-dim quantum states.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

- multiple qubit states are described by tensor product.

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$

- n-qubit states are equivalent to 2^n-dim vectors.

$$|\psi\rangle = \sum_{n=0}^{2^n-1} \psi_n |n\rangle = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{2^n-1} \end{bmatrix}$$

Unitary gate

- Quantum states are evolved under unitary operations.
- n-qubit unitary gates are $2^n \times 2^n$ unitary matrices.

$$\begin{bmatrix} \psi'_0 \\ \psi'_1 \\ \vdots \\ \psi'_{2^n-1} \end{bmatrix} = \begin{bmatrix} & & \\ & U & \\ & & \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{2^n-1} \end{bmatrix}$$

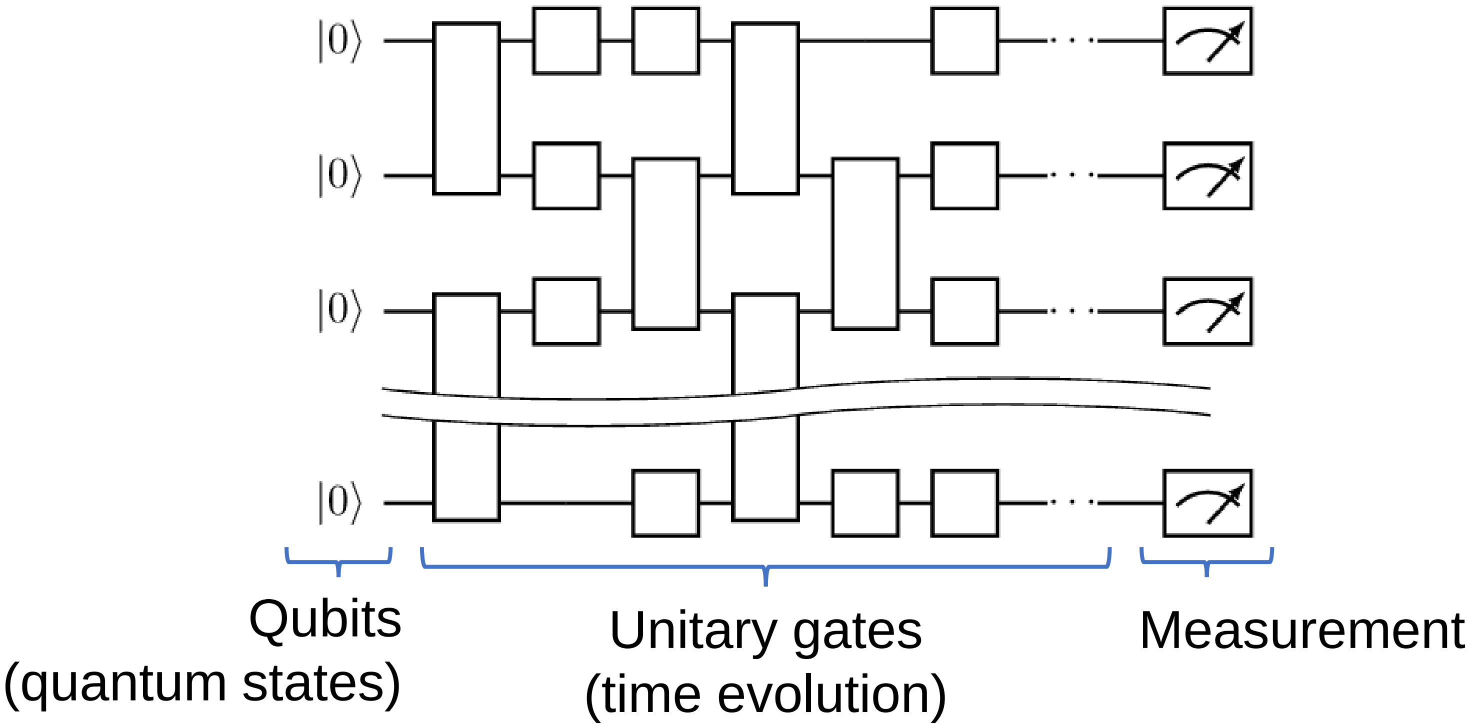
- $\{H, \mathrm{CNOT}, T\}$ gate set is universal and usually used.

Measurement

- We have to measure the evolved state to extract information (solution).
- Measurement protocol performs the task efficiently.
  - ✓ Which qubits?
  - ✓ Which basis?
  - ✓ How many samples?

Introduction to quantum computers

Quantum circuits – diagrams of quantum computing procedures



Quantum algorithms ≈ Constructing a quantum circuit to solve a given problem

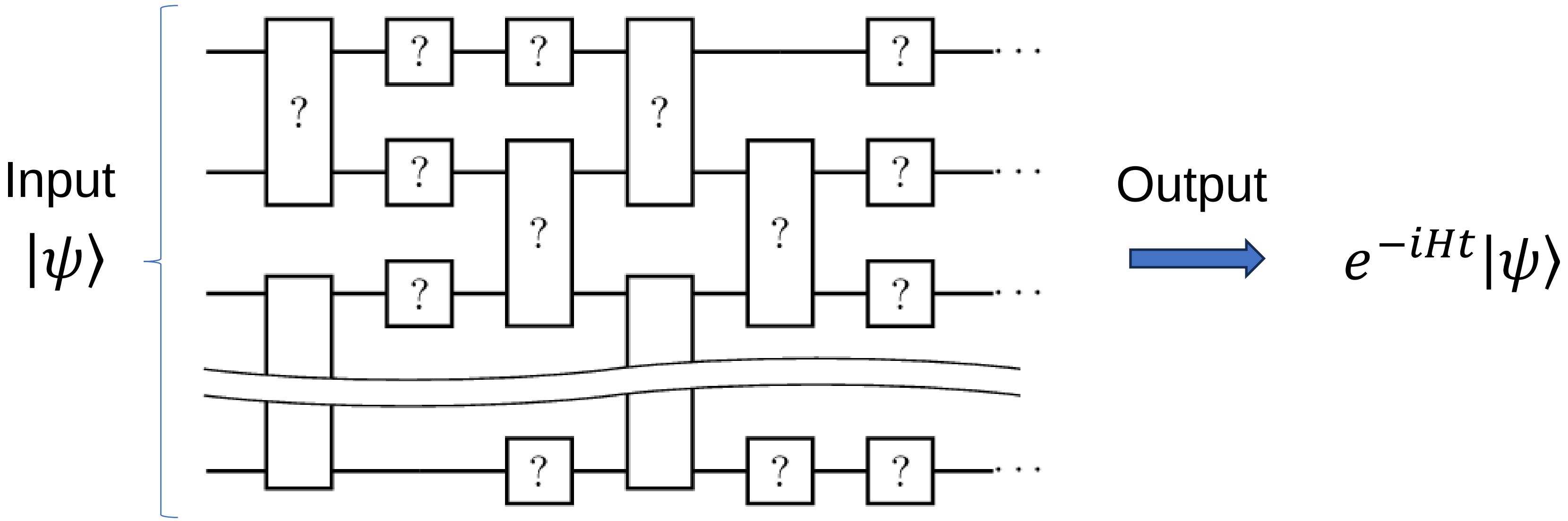
Hamiltonian simulation

- Hamiltonian simulation is one of the promising task with **exponential quantum advantage**.
- Problem statement:

S. Lloyd, Science, (1996).

Input: a Hamiltonian H and an initial quantum state $|\psi\rangle$.

Output: the evolved quantum state $e^{-iHt}|\psi\rangle$.



How to construct a quantum circuit for simulating a Hamiltonian?

Trotter formula (Conventional method)

S. Lloyd, Science, (1996).

Divide the whole evolution into short time evolution of each term

- Given a Hamiltonian $A + B$, we can implement $e^{-i(A+B)t}$ if we have e^{-iAt} and e^{-iBt}.

- (first-order) Trotter formula:

$$e^{-i(A+B)t} = \left(e^{-iAt/r}e^{-iBt/r}\right)^r + O\left(\frac{\|[A,B]\|t^2}{r}\right)$$

A. M. Childs, et al. Phys. Rev. X, (2021).

- To achieve the total error ε, we set

$$r = O\left(\frac{\|[A,B]\|t^2}{\varepsilon}\right)$$

Trotter formula (Conventional method)

S. Lloyd, Science, (1996).

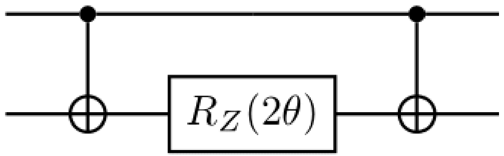
Example: transverse field Ising model

$$H_{TIM} = \sum_{i,j=1}^{n} J_{ij} Z_i Z_j + \sum_{i=1}^{n} h_i X_i$$

- Trotter formula implies that we only need to implement

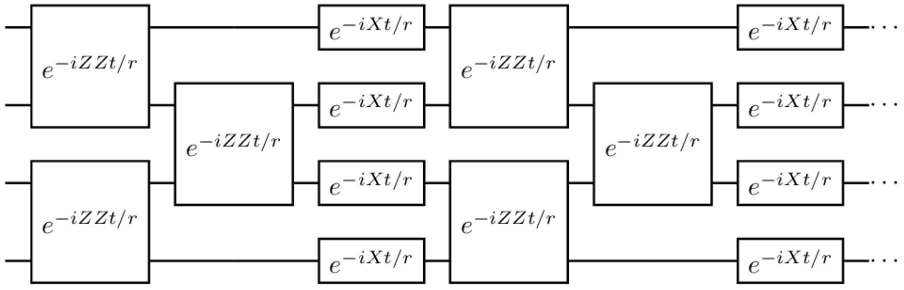
$$e^{-iX_i\theta} \text{ and } e^{-iZ_iZ_j\theta}$$

just a Pauli X rotation gate



$$= e^{-iZ_iZ_j\theta}$$

$$e^{-iH_{TIM}t} \approx$$



$e^{-i\sum_{ij} J_{ij}Z_iZ_jt/r}$ $e^{-i\sum_i h_iX_it/r}$ Repeat r times

Quantum singular value transformation (Recent method)

A. Gilyen, et al, STOC (2019)

- Trotter formula achieves the complexity of $O(\|[A,B]\|t^2/\varepsilon)$.

Can we further improve this complexity?

Yes!

- Quantum singular value transformation (QSVT) achieves $O\big(\|H\|t + \log(1/\varepsilon)\big)$.
  - Provably optimal complexity in terms of $\|H\|, t, 1/\varepsilon$.

A. Gilyen, et al, STOC (2019)

- QSVT implements a polynomial transformation of any matrix.
  - Today we focus on a Hermitian matrix H.

Quantum singular value transformation (Recent method)

A. Gilyen, et al, STOC (2019)

- **Block-encoding:** embed a Hamiltonian H into a unitary operator U.

S. Chakraborty, et al, ICALP, (2019)

$$U = |0^b\rangle\langle 0^b| \otimes H + \cdots = \begin{bmatrix} H & \cdot \\ \cdot & \cdot \end{bmatrix}$$

- Let parametrized reflection operator $R(\phi)$ as:

$$R(\phi) = e^{i\phi}|0^b\rangle\langle 0^b| \otimes I + e^{-i\phi}\left(I - |0^b\rangle\langle 0^b|\right) \otimes I$$

  - This operator can be implemented using multi-controlled gates.

- Surprisingly, a sequence of U and $R(\phi)$ provides a polynomial transformation of H.

$$R(\phi_d)U \cdots R(\phi_2)U^\dagger R(\phi_1)UR(\phi_0) = \begin{bmatrix} P(H) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

d: degree of polynomial P

- QSVT with polynomial approximation of e^{-iHt} provides a new Hamiltonian simulation method.

$$e^{-iHt} \approx \sum_{k=0}^{d} \beta_k H^k$$

with $d = O\left(\|H\|t + \log(1/\varepsilon)\right)$

Block-encoding

Example: linear combination of unitary operators

$$H = \sum_{l=0}^{L-1} \alpha_l U_l$$

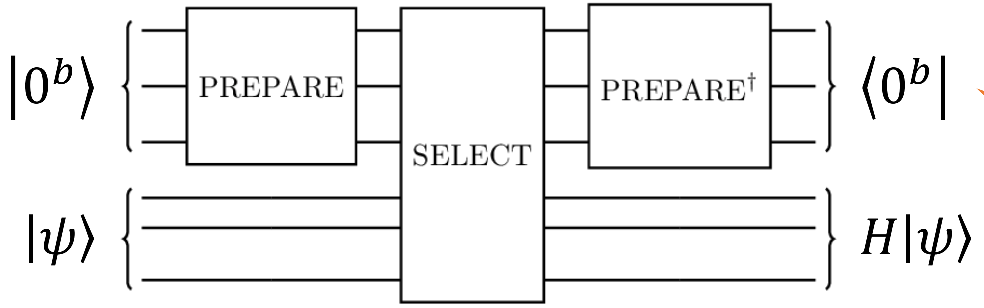
- Assume $a_l > 0$ and $\sum_l a_l = 1$.

➢ Let PREPARE operator and SELECT operator as:

$$\text{PREPARE}\,|0^b\rangle = \sum_{l=0}^{L-1} \sqrt{\alpha_l}\,|l\rangle \qquad \text{SELECT} = \sum_{l=0}^{L-1} |l\rangle\langle l| \otimes U_l$$

➢ Block-encoding circuit:

R. Babbush, et al. Phys. Rev. X, (2018).



When the measurement outcome is $|0^b\rangle$

➢ PREPARE and SELECT can be implemented with $O(L)$ getes.

Summary – Hamiltonian simulation

- Trotter formula $e^{-i(A+B)t} \approx \left(e^{-iAt/r} e^{-iBt/r}\right)^r$
  - The complexity of $O(t^2/\varepsilon)$
  - Simple implementation
- Quantum singular value transformation (QSVT) $e^{-iHt} \approx \sum_{k=0}^{d} \beta_k H^k$
  - The optimal complexity of $O\left(t + \log(1/\varepsilon)\right)$
  - Applicable to a block-encoded Hamiltonian

$$R(\phi_d) U \cdots R(\phi_2) U^\dagger R(\phi_1) U R(\phi_0) = \begin{bmatrix} e^{-iHt} & \cdot \\ \cdot & \cdot \end{bmatrix}$$

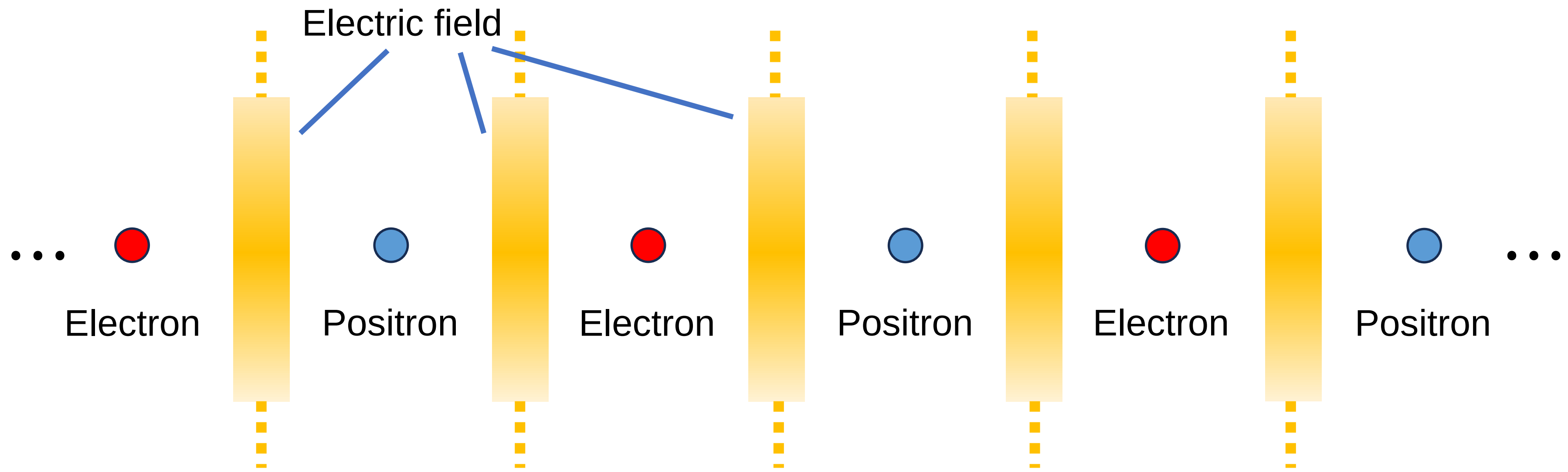
Efficient implementation of block-encoding is important!

Our recent work:
efficient quantum algorithm for simulating the Schwinger model

Kazuki Sakamoto, Hayata Morisaki, Junichi Haruna, Etsuko Itou, Keisuke Fujii, Kosuke Mitarai,
“End-to-end complexity for simulating the Schwinger model on quantum computers”,
Quantum 8, 1474 (2024), arXiv:2311.17388

Schwinger model

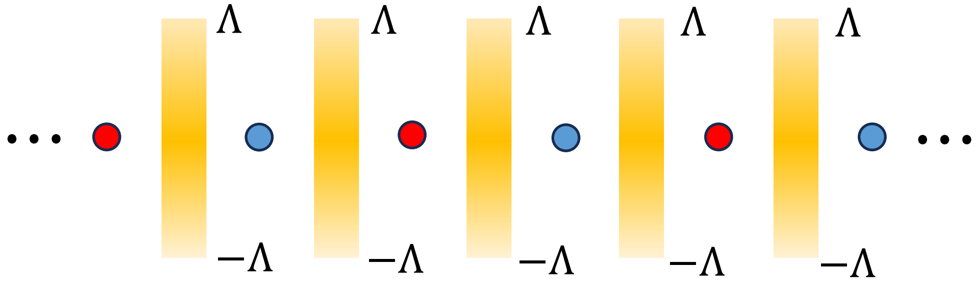
- one of the simplest yet non-trivial gauge theories



Two types of Hamiltonian formulation

Truncate the electric field at Λ

J. Kogut, et al, Phys. Rev. D (1975).



Remove the electric field with Gauss's law

T. Banks, et al, Phys. Rev. D (1976).

Our work

Schwinger model

Previous works (real time evolution e^{-iHt})

- System size : N
- Precision : ε
- Evolution time : t

- The Hamiltonian formulation without electric field

  E. A. Martinez, et al, Nature 534, 516 (2016).
  N. H. Nguyen, et al, PRX Quantum 3, 020324 (2022). ········ $O(N^{4.5}t^{1.5}/\varepsilon^{0.5})$

  - Based on Trotter formula

  Our work ········ $\tilde{O}\left(N^4 t + \log(1/\varepsilon)\right)$

- The Hamiltonian formulation which includes electric field

  A. F. Shaw, et al, Quantum 4, 306 (2020). ········ $\tilde{O}(N^{2.5}t^{1.5}/\varepsilon^{0.5}))$

  - Based on Trotter formula
  - Provides rigorous cost analysis

  Y. Tong, et al, Quantum 6, 816 (2022). ········ $\tilde{O}(Nt\ \mathrm{polylog}(1/\varepsilon))$

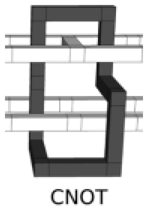
  - The smallest query complexity at present
  - Needs a huge number of qubits

Rigorous resource estimates without scaling are needed for a comparison.

Runtime is estimated via the number of T gates

T gate is costly in FTQC due to error correction

- An arbitrary unitary operator can be decomposed into Clifford + T gates.
- Clifford gates are easy, but T gates need large space-time overhead.



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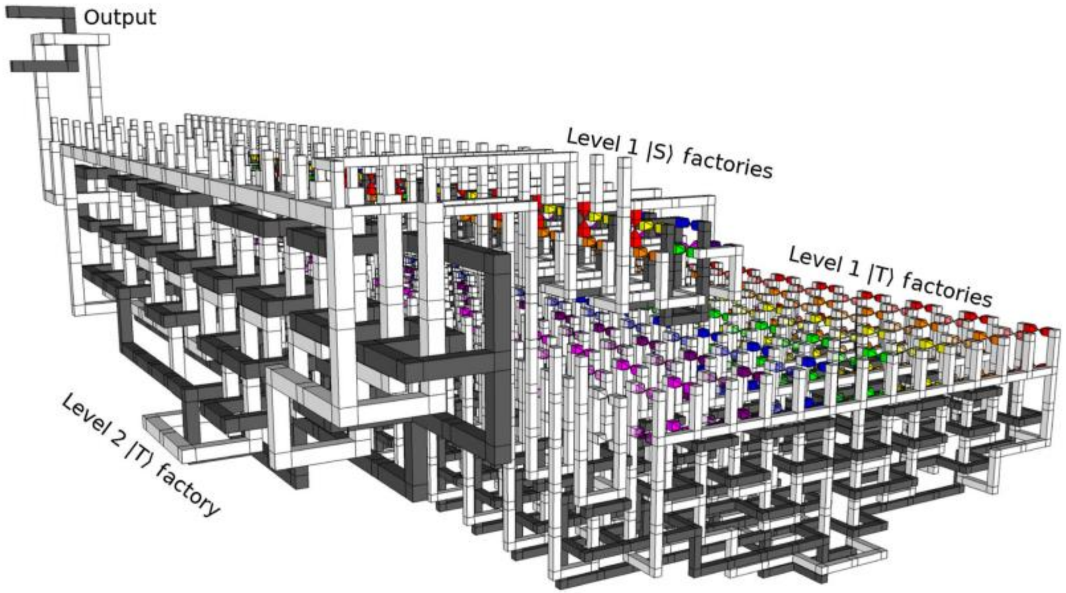


R. Babbush, et al. Phys. Rev. X, (2018).

The number of T gates dominates actual runtime.

Result – efficient block-encoding

- The Schwinger model Hamiltonian after Jordan-Wigner transformation:

$$H_S = J\sum_{n=0}^{N-2}\left(\sum_{i=0}^{n}\frac{Z_i+(-1)^i}{2}+\frac{\theta_0}{2\pi}\right)^2+\frac{w}{2}\sum_{n=0}^{N-2}(X_nX_{n+1}+Y_nY_{n+1})+\frac{m}{2}\sum_{n=0}^{N-1}(-1)^nZ_n$$

- Naively, we need $O(N^2)$ T gates for the block-encoding since H_S has $O(N^2)$ terms.

- **Our implementation requires only $O(N)$ T gates.**
  - Decompose the Hamiltonian into several parts as below:

$$H_S = \boxed{\frac{J}{4}\sum_{n=1}^{N-1}\left(\sum_{i=0}^{n-1}Z_i\right)^2}+\boxed{J\frac{\theta}{2\pi}\sum_{\substack{n=1\\n:\text{even}}}^{N-1}\sum_{i=0}^{n-1}Z_i}+\boxed{J\left(\frac{1}{2}+\frac{\theta}{2\pi}\right)\sum_{\substack{n=1\\n:\text{odd}}}^{N-1}\sum_{i=0}^{n-1}Z_i}+\boxed{\frac{w}{2}\sum_{n=0}^{N-2}X_nX_{n+1}}+\boxed{\frac{w}{2}\sum_{n=0}^{N-2}Y_nY_{n+1}}+\boxed{\frac{m}{2}\sum_{n=0}^{N-1}(-1)^nZ_n}$$

  - Uniform superposition states $\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle$ can be prepared efficiently with $O(\log N)$ T gates.

Y. R. Sanders, et al, PRX Quantum (2020).

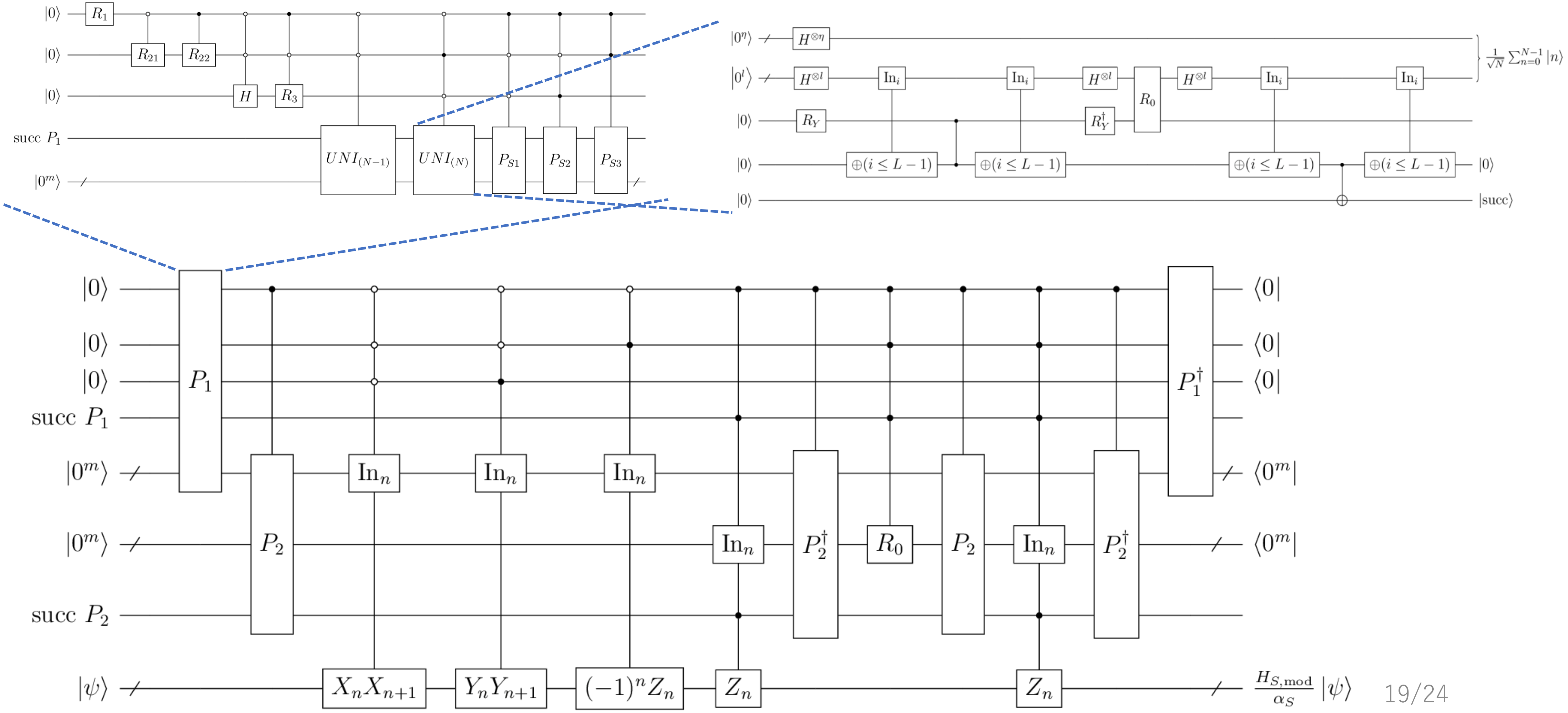
⟹ Leading to efficient block-encodings of each term

  - Take a linear combination of the block-encodings of each term

A. Gilyen, et al, STOC (2019)

Result – efficient block-encoding

Overall circuit description



Result – resource estimates

Simulating the creation and annihilation of particle pairs

- The ground state of H_S for $J = \theta_0 = w = 0, m = m_0$ is the vacuum state.

$$|\mathrm{vac}\rangle = |1010 \ldots\rangle$$

  - The state without any particle.

- Evolve $|\mathrm{vac}\rangle$ under H_S for time t and estimate the amplitude of $|\mathrm{vac}\rangle$.

$$\left|\langle \mathrm{vac}| e^{-iH_S t} |\mathrm{vac}\rangle\right|$$ ：vacuum persistence amplitude J. Schwinger, Phys. Rev. (1951).

  - Simulation of the creation and annihilation of electron-positron pairs.

Based on our block-encoding,
how much resource is required to compute this quantity?

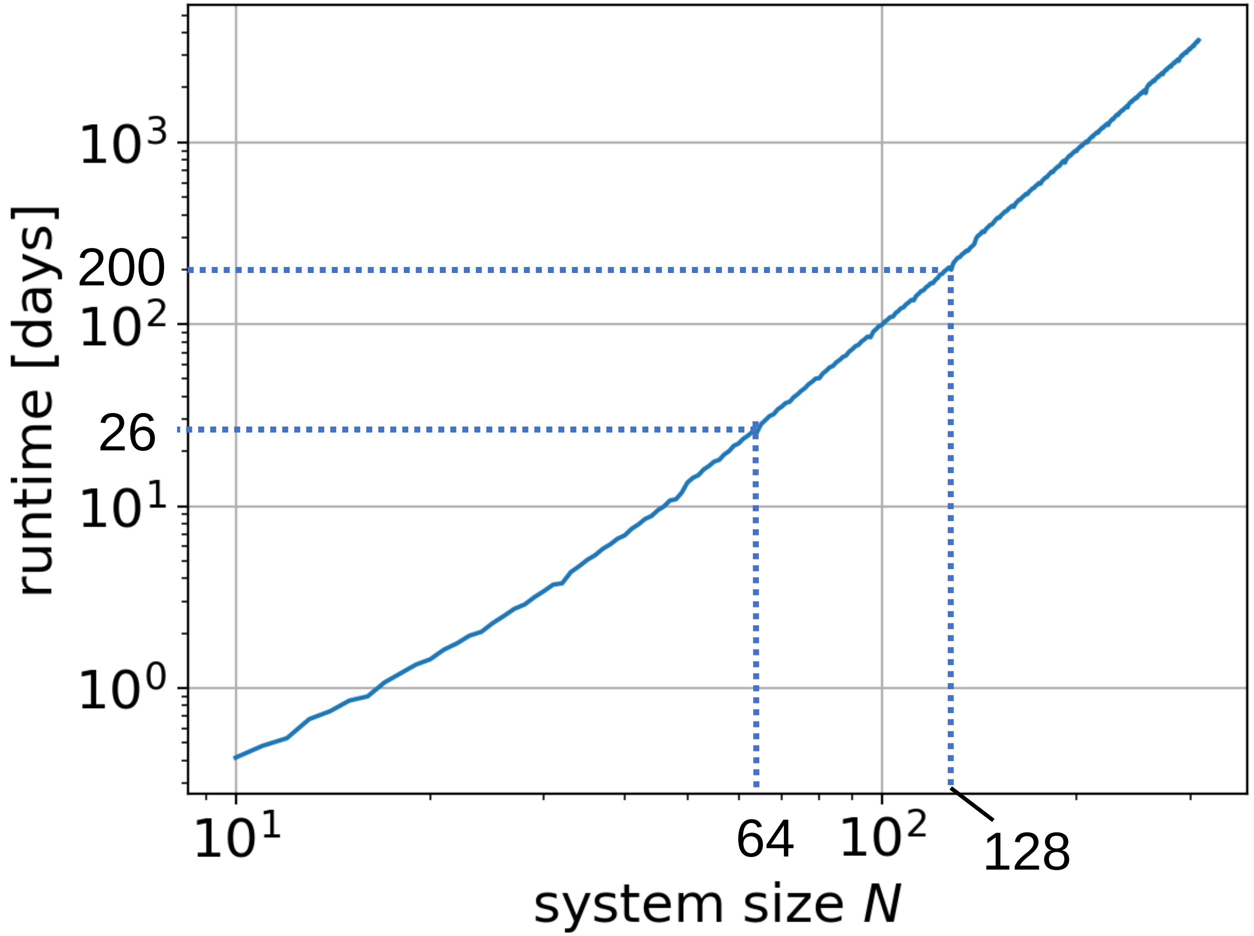
Result – resource estimates

Runtime

- Parameters
  - Precision (additive error) : $\varepsilon = 0.01$
  - Evolution time : $t = 4$
  - T gate consumption rate : 1MHz
  - Lattice spacing : $a = 0.2$
  - electron mass : $m = 0.1$
  - $w = \frac{1}{2a} = 2.5$
  - $J = \frac{g^2 a}{2} = 0.1, \ (g = 1)$
  - $\theta_0 = \pi$

Examples

System size	Runtime [days]
64	26
128	200



Runtime for calculating the vacuum persistence amplitude.

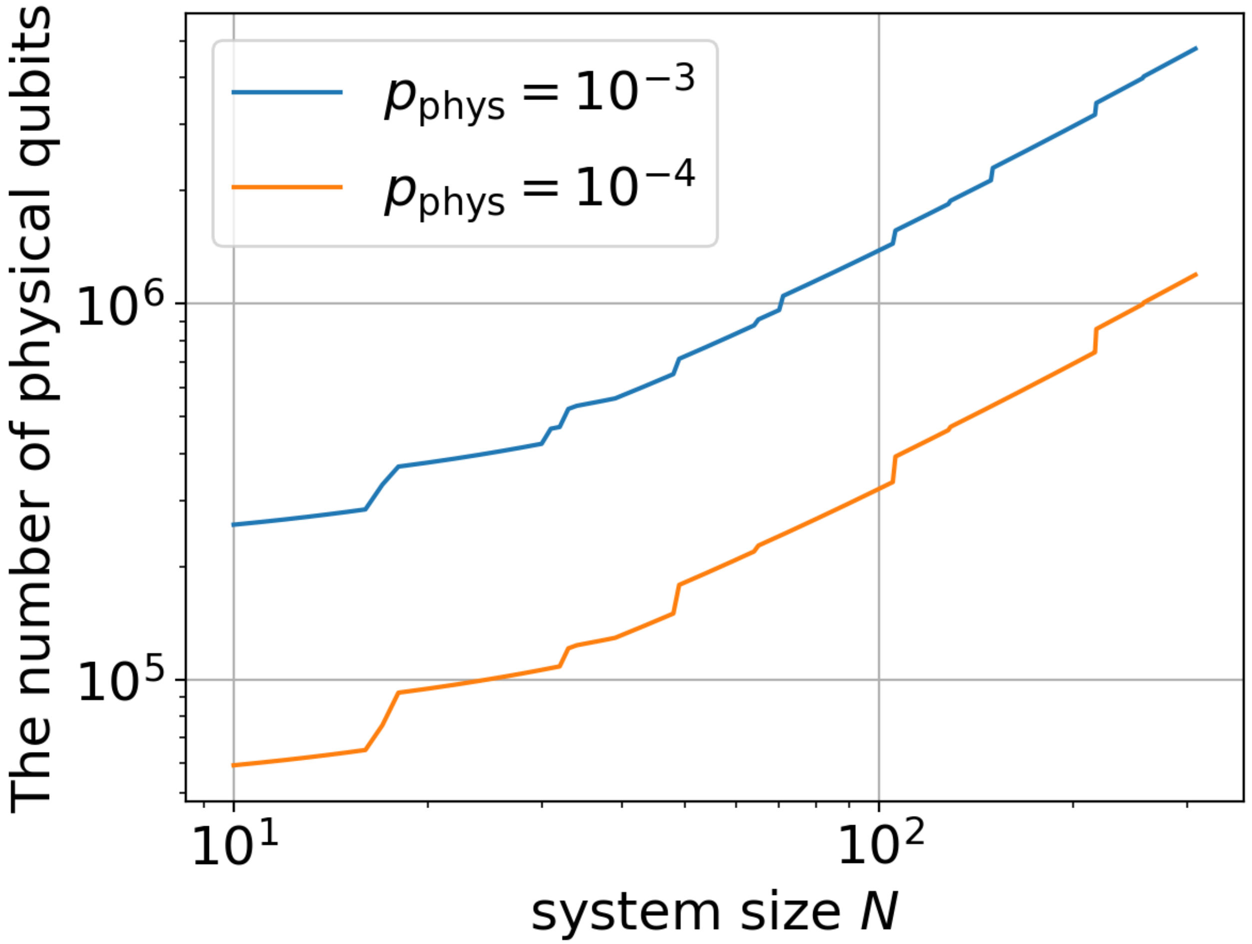
Result – resource estimates

The number of physical qubits

- Parameters
  - Precision (additive error) : $\varepsilon = 0.01$
  - Evolution time : $t = 4$
  - Lattice spacing : $a = 0.2$
  - electron mass : $m = 0.1$
  - $w = \frac{1}{2a} = 2.5$
  - $J = \frac{g^2 a}{2} = 0.1,\ (g = 1)$
  - $\theta_0 = \pi$

Examples ($N = 64$)

Physical error rate	Physical pubits
10^{-3}	9×10^5
10^{-4}	2×10^5



The number of physical qubits
for calculating the vacuum persistence amplitude.

Result – resource estimates

Comparison with the previous work (real time evolution e^{-iHt})

A. F. Shaw, et al, Quantum (2020)

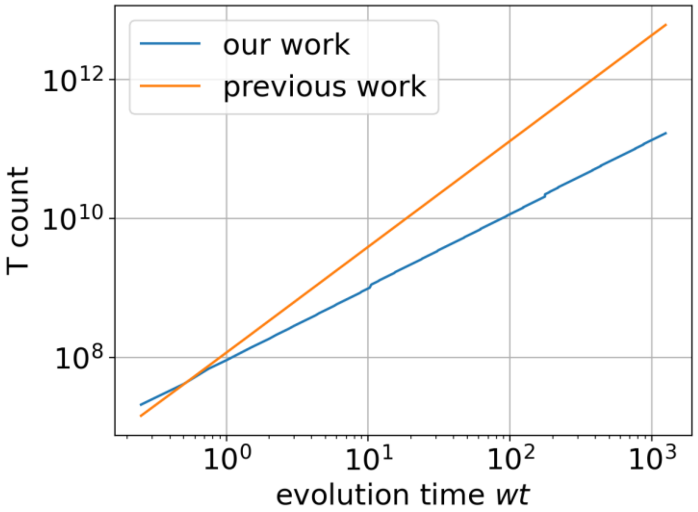
Our work (QSVT) : $\tilde{O}(N^4 t + \log(1/\varepsilon))$

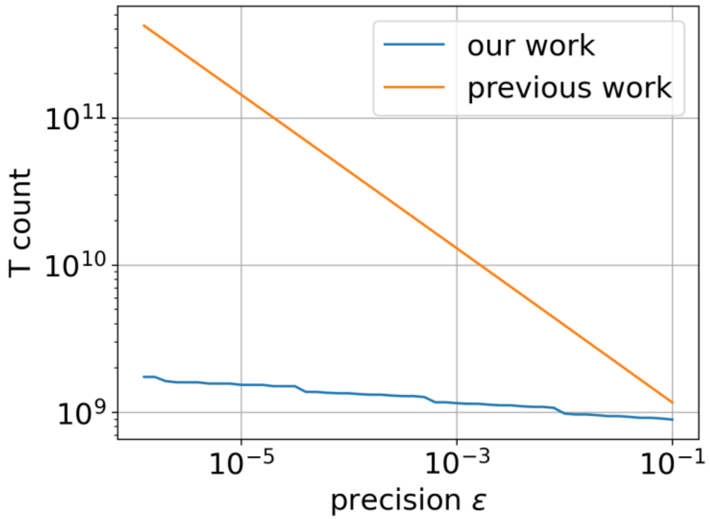
The previous work (Trotter)
: $\tilde{O}(N^{2.5} t^{1.5} / \varepsilon^{0.5})$

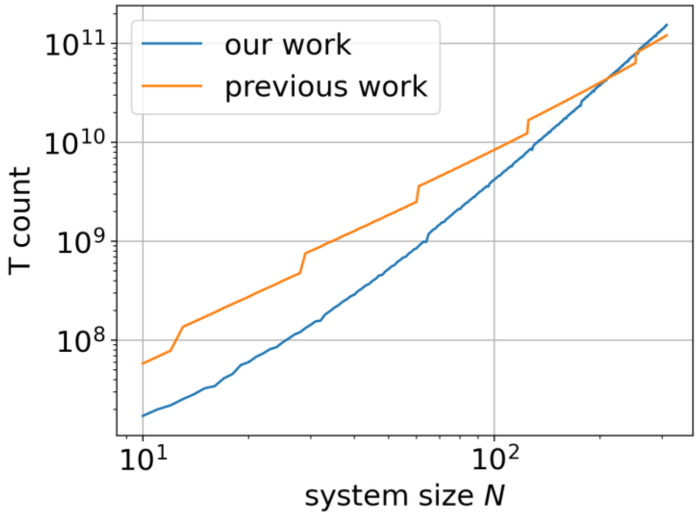
domain	Which is better?
Long-time	Our work
High-precision	Our work
Large-system	Previous work

- Qubit requirement

Our work		Previous work
$N + O(\log N)$	$\ll$	$O(N \log N)$







- Parameters
  - Precision (additive error) : $\varepsilon = 0.01$
  - Evolution time : $t = 4$
  - Lattice spacing : $a = 0.2$
  - Electron mass : $m = 0.1$
  - $w = \frac{1}{2a} = 2.5$
  - $J = \frac{g^2 a}{2} = 0.1, (g = 1)$
  - $\theta_0 = \pi$

Conclusion

- Comparison to other physical models

Condensed matter physics (e.g. Hubbard model)	$<$	Schwinger model	$\approx$	Quantum chemistry (electronic Hamiltonian)
N. Yoshioka, et al, npj Quantum Information 4, 45 (2024)				J. Lee, et al, PRX Quantum (2021)
T count : $\sim 10^8$		$\sim 10^{12}$		$\sim 10^{12}$

Summary

- An efficient block-encoding of the Schwinger model Hamiltonian
  - Decompose the Hamiltonian into several parts.
  - Use $O(\log^2 N)$ T gates for P, $O(N)$ T gates for V, with a normalization factor of $O(N^3)$.

- End-to-end complexity for the Schwinger model
  - Estimate the vacuum persistence amplitude.
  - The T gate speed of 1 MHz is minimum requirement.

K. Sakamoto, et. al. Quantum 8, 1474 (2024)

Appendix