Numerical analysis of entanglement entropy using tensor network

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in collaboration with

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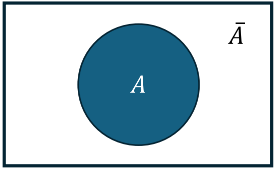
Quantum entanglement

- A correlation between two subregions of quantum many-body systems.
- Entanglement entropy (EE) is a measure of the degree of quantum entanglement.

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$



where ρ_A is a reduced density matrix of the subregion A and given by $\rho_A = \mathrm{tr}_{\bar{A}}\rho$.

Entanglement entropy

- Entanglement entropy has many applications in various fields:
  - Particle physics: blackhole entropy, quantum order parameter
  - Quantum information: quantum computer, quantum teleportation
  - Condensed matter physics: phase structure of metallic condensed matter system

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$\rightarrow$We focus on determining the critical temperature of a (1+1)D lattice model using the EE.

Entanglement entropy and system detail

The subregion size l dependence of the EE S_A tells us the detail of the system.

- Entropic c-function $C(l)$, l: length of the subregion A

$$C(l) = \frac{l}{2} \frac{\partial S_A}{\partial l}$$

$C(l)$ monotonically decreases along the RG flow

$\rightarrow$ detail of the effective degrees of freedom can be extracted.

- The EE on the quantum critical point is given by

$$S(l) = \frac{c}{3} \log l + k,$$

where c is a central charge and k is a constant.

$\rightarrow$ central charge can be extracted.

Numerical analysis of entanglement entropy

- Monte Carlo method
  - calculates the entropic c-function.
    e.g. 4D SU(3) gauge theory [Itou-Nagata-Nakagawa-Nakamura-Zakharov, 2015]
  - based on the definition of the EE on lattice
    [Aoki-Iritani-Nozaki-Numasawa-Shiba-Tasaki, 2015],
  - has sign problem.

Numerical analysis of entanglement entropy

- Tensor network method (In this talk, we focus on those of Lagrangian formalism)
  - has no sign problem.
  - directly computes the reduced density matrix and the EE.
    e.g. (1+1)D O(3) non-linear sigma model [Kuramashi-Luo, 2023]
  - is limited to the case of half-space subregion A.

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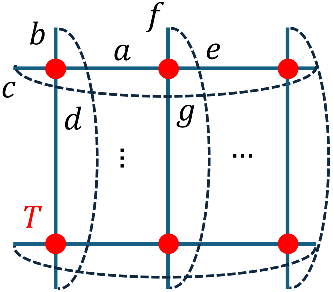
We compute the subregion size dependence of the EE with our new method [Hayazaki-Kadoh-Takeda-GT, work in progress].

Tensor network (Lagrangian formalism)

- Partition function can be directly computed.

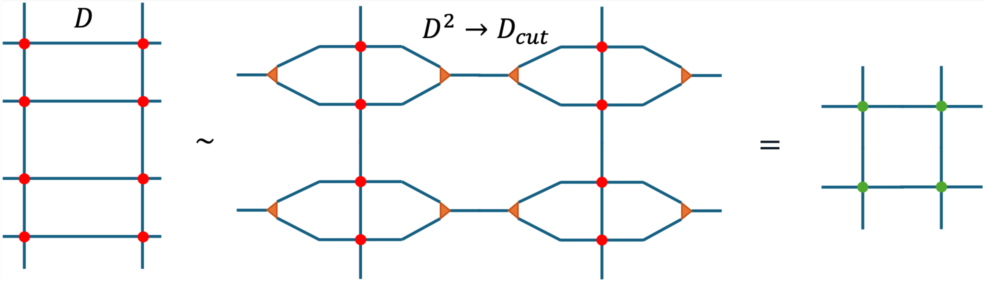
$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_{\ldots,a,b,c,d,e,f,g,\ldots} \ldots T_{abcd} T_{efag} \ldots =$$



- We need some "coarse-graining" to reduce the computational cost.

Tensor Renormalization group (TRG)

- Recursively approximates multiple tensors as one tensor.
  e.g. Higher-order TRG (HOTRG) algorithm

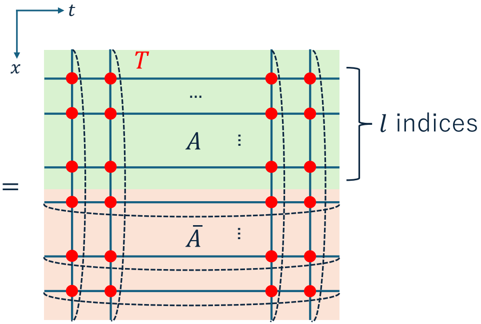


- Various TRG algorithms are proposed:
  A-TRG [Adachi-Okubo-Todo, 2019, 2019], Triad-TRG [Kadoh-Nakayama, 2019], etc.

Tensor network computation of reduced density matrix

Reduced density matrix ρ_A of the subregion A with spatial size l:

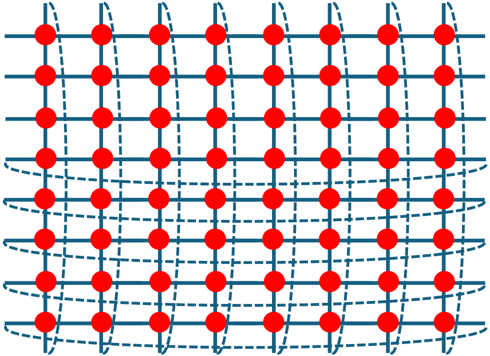
$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho =$$



We simplify this network using HOTRG algorithm.

Tensor network computation of reduced density matrix

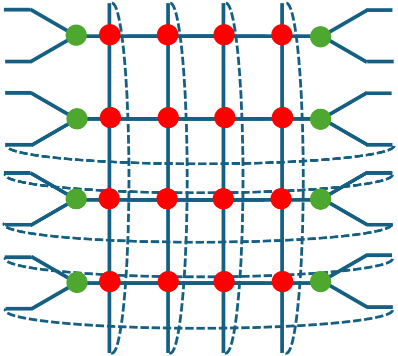
Example: total spatial size is 8 and subregion size 3.



Tensor network representation of the reduced density matrix before coarse-graining.

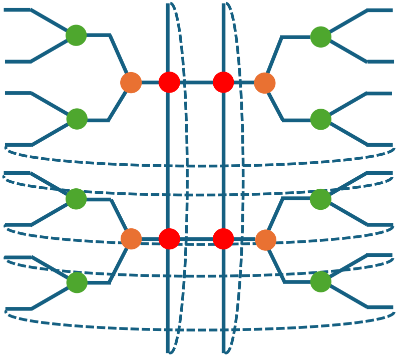
Tensor network computation of reduced density matrix

After one HOTRG coarse-graining procedure:



Tensor network computation of reduced density matrix

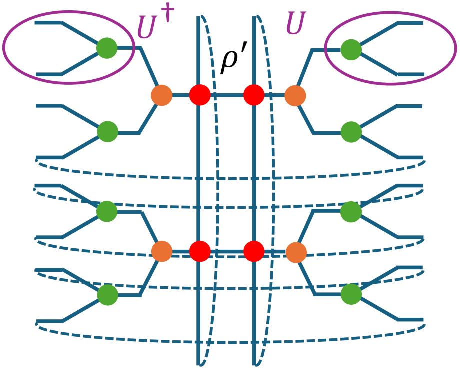
After two HOTRG coarse-graining procedures:



At this stage, we can simplify this network further!

Tensor network computation of reduced density matrix

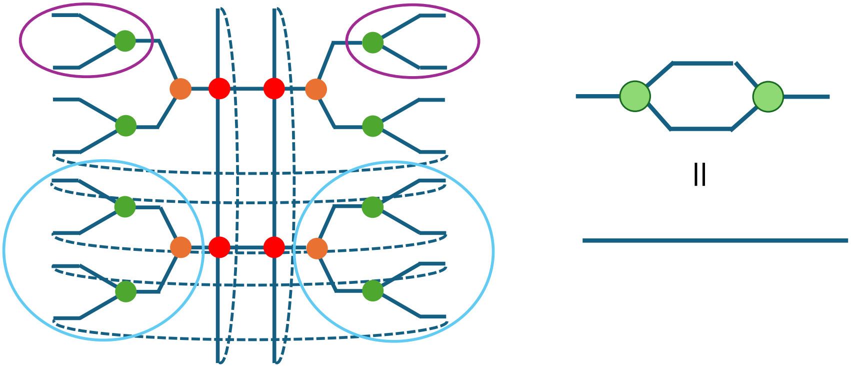
Tensors U and $U^\dagger$ do not contribute to the entanglement entropy.



$$S_A = -\mathrm{tr}\rho_A \log \rho_A = -\mathrm{tr}U^\dagger \rho'_A U \log(U^\dagger \rho'_A U) = -\mathrm{tr}\rho'_A \log \rho'_A$$

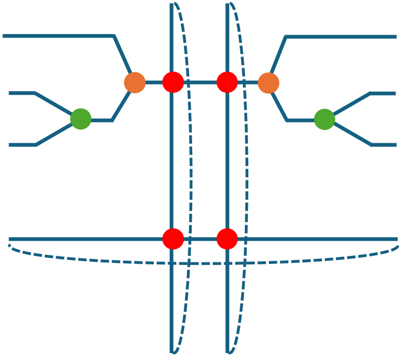
Tensor network computation of reduced density matrix

Some isometry tensors can be contracted and become an identity matrix.



Tensor network computation of reduced density matrix

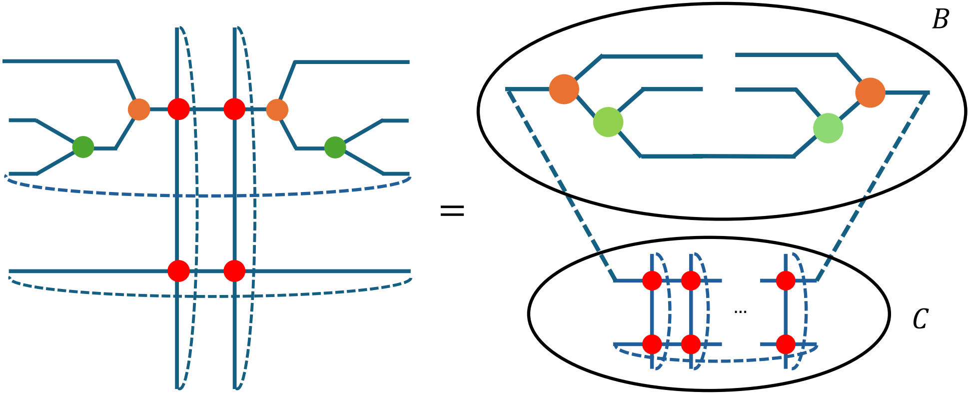
Finally, we obtain the simplified tensor network of the reduced density matrix below:



We established the algorithm to obtain this final result directly.

Our algorithm

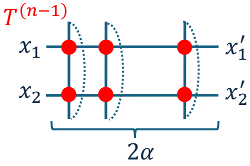
The simplified tensor network of the reduced density matrix consists of two parts: core matrix C and boundary factor B.

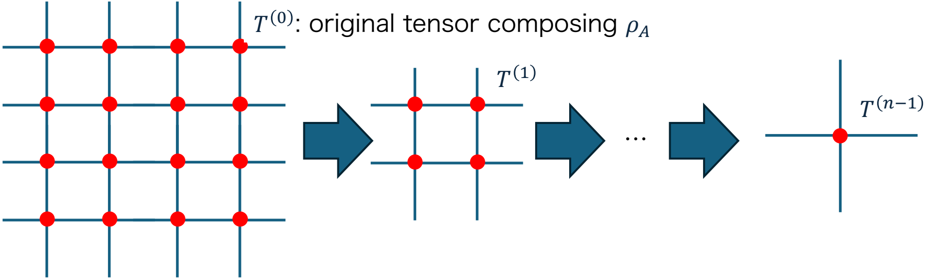


Our algorithm

In the following, we set the total spatial size $L_x = 2^n$, temporal size $L_t = \alpha \cdot 2^n$.
The core matrix C consists of coarse-grained tensor $T^{(n-1)}$.

$C_{(x_1 x_2)(x'_1 x'_2)} =$

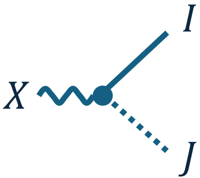




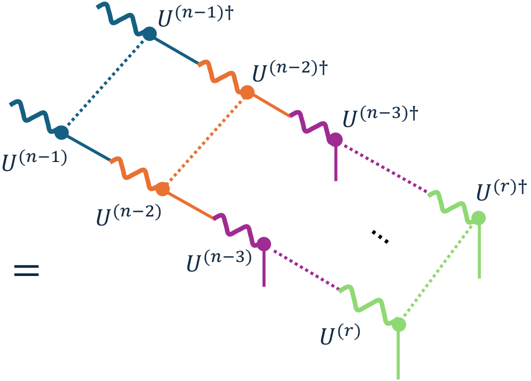
Our algorithm

The boundary factor B consists of isometry tensors $U^{(i)}$ and $U^{(i)\dagger}$ obtained in the coarse-graining procedure of tensor $T^{(n-1)}$

$U^{(k)}_{(IJ)X} =$



$B =$



The contraction of isometry tensors depends on the subregion size l.

Numerical Analysis: (1+1)D XY model

- Partition function and action:

$$Z = \int \prod_{x=0}^{L_x} \prod_{t=0}^{L_t} \frac{d\theta_{x,t}}{2\pi} e^{-S}$$

$$S = -\beta \sum_{x,t} \cos(\theta_{x,t+1} - \theta_{x,t}) - \beta \sum_{x,t} \cos(\theta_{x+1,t} - \theta_{x,t})$$

β: inverse temperature

Spatial lattice size: 1024, temporal lattice size: $2^8 \times 1024$

- XY model exhibits the topological BKT phase transition at $T = T_{\mathsf{BKT}}$, and $0 < T < T_{\mathsf{BKT}}$ is the critical line. ($T_{\mathsf{BKT}} = 0.892943(2)$ [Ueda-Oshikawa, 2021])

Tensor network for XY model

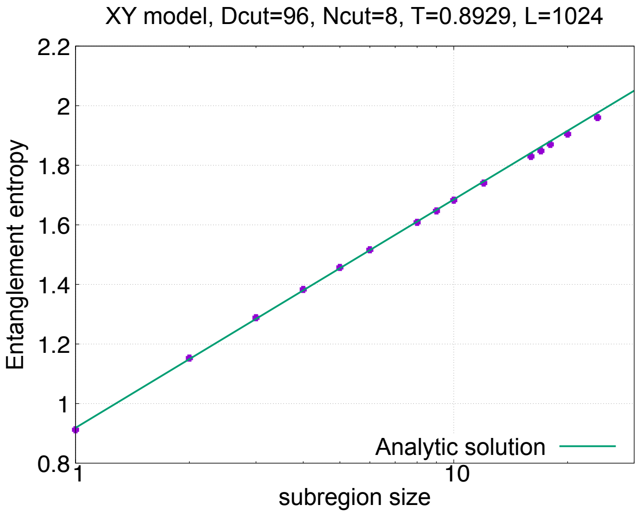
Partition function$Z$:

$$Z = \int \prod_{x=0}^{L_x} \prod_{t=0}^{L_t} \frac{d\theta_{x,t}}{2\pi} e^{-S} = \prod_{\text{lattice}} T_{xx'yy'}$$

$$T_{xx'yy'} \equiv \sqrt{e^{(y+y')\mu}} \delta_{x'+y'-x-y} \sqrt{I_{y'}(\beta)} \sqrt{I_y(\beta)} \sqrt{I_{x'}(\beta)} \sqrt{I_x(\beta)}$$

$I_x(\beta)$: modified Bessel function of the first kind, where x takes from $-\infty$ to ∞.
$\rightarrow$ We regularize $I_x(\beta)$ by introducing the cutoff N_{cut}: $-N_{\text{cut}} \leq x \leq N_{\text{cut}}$

Result - subregion size dependence of EE and central charge



- subregion size l:
  $l = 2^p + 2^q \ (q < p)$
- Analytic solution of EE of finite size subregion

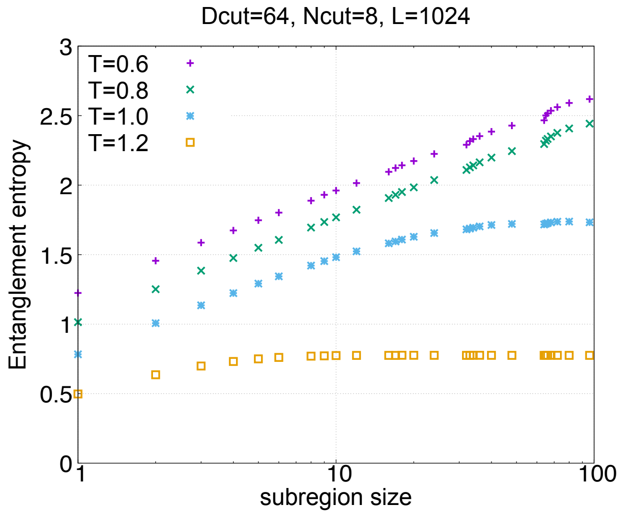
$$S(l, L) = \frac{c}{3} \log\left(\sin\left(\frac{\pi l}{L}\right)\right) + k$$

- Central charge c by fitting the result to the analytic solution

$$c = 0.998(5)$$

$\rightarrow$agrees with known result $c = 1$.

Result - temperature dependence of EE



- On the critical line $T = 0.6, 0.8 < T_{BKT}$

$$S(l, L) = \frac{c}{3} \log \left(\sin \left(\frac{\pi l}{L} \right) \right) + k$$
$$\sim \frac{c}{3} \log l + \text{const.}$$

- Non-critical $T = 1.0, 1.2 > T_{\text{BKT}}$: l dependence for small l $\because$ finite correlation length.
- We may determine the transition temperature using the EE.

Conclusion and discusson

Summary of this talk:

- We studied the subregion size dependence of the entanglement entropy in the 1+1D XY model.
- We determined the central charge on the critical line $T < T_{\mathsf{BKT}}$ using the subregion size dependence of the EE.
- Difference in the behavior of the EE implies that we can determine the transition temperature using the EE.

Conclusion and discusson

Future direction:

- Compute entanglement entropy of a larger subregion size.
- Determine transition temperature.

Method:

- More efficient TRG algorithm
  e.g. HOSRG [Z. Y. Xie, et al., 2012]
- Parallelization of algorithm
  e.g. Parallelized HOTRG [Yamashita-Sakurai, 2021]

Backup - Determining the fitting range

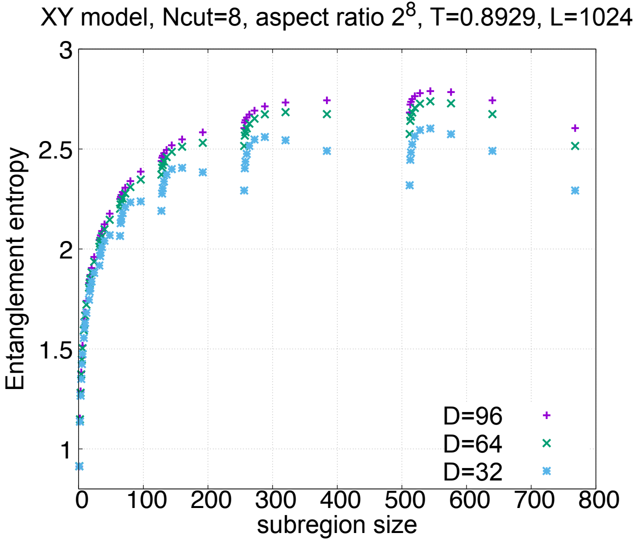
1. Compute the EE $S(l'), S(l)$ and obtain the central charge $c(l)$ where $l = 2^p, l' = 2^{p+1}$ or $l = 2^p + 2^q, l' = 2^{p+1} + 2^q$.

$$
\begin{aligned}
c(l) =& 3\,\frac{S_A(l') - S_A(l)}{\log\sin\frac{l'\pi}{1024} - \log\sin\frac{l\pi}{1024}} \\
c(1) =& 1.041435054 \\
c(2) =& 0.997233754, \ldots
\end{aligned}
$$

2. Determine the fitting range as 1 to x, where x is the smallest inetger that satisfies

$$|c(x) - 1| \geq |c(1) - 1| = 0.041435054.$$

Backup - Dcut dependence of the EE



Backup - Boundary factor

The boundary factor B is composed of isometries $U^{(n-2)}, U^{(n-3)}, \dots, U^{(r)}$.
The integer r is the largest one that satisfies $a_k \neq b_k$, where

$$l = \sum_{k=0}^{n-1} a_k 2^k \ (a_k = 0, 1),$$
$$l - 1 = \sum_{k=0}^{n-1} b_k 2^k \ (b_k = 0, 1).$$

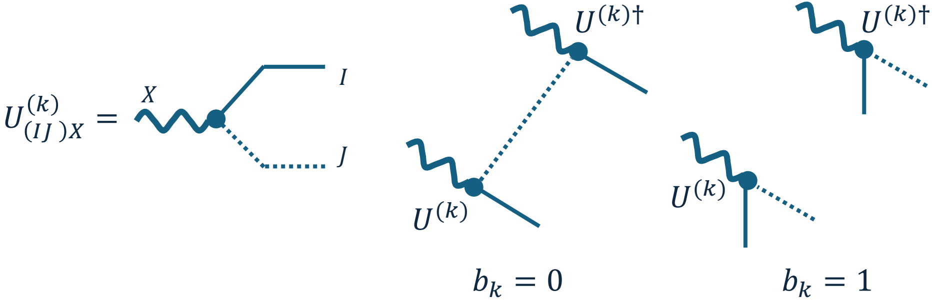
For example, letting $L = 2^4$ and $l = 5$, we have

$$\begin{aligned} l =& 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0, \\ l - 1 =& 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0, \end{aligned}$$

and $r = 0$.

Backup - Boundary factor

b_k determines the form of contraction of isometry $U^{(k)}$ and $U^{\dagger(k)}$.

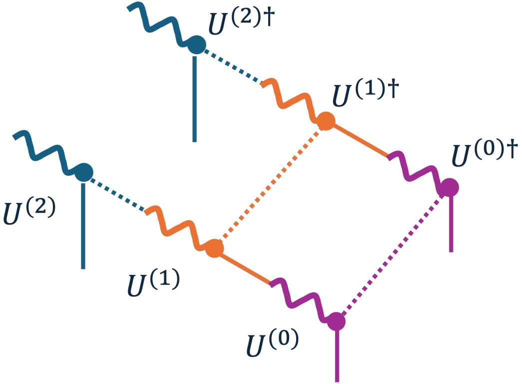


The index of $U^{(k)}$ represented by a wavy line is contracted with the index of $U^{(k+1)}$ represented by a solid line or a dotted line.

Backup - Boundary factor

Example: Total spacial size 16, and subregion size 5.

$\rightarrow b_2 = 1, b_1 = 0, b_0 = 0$ and $r = 0$.



The indices represented by a wavy line are contracted with core matrix.