# Entanglement Filtering in 3D Tensor-Network Renormalization Group

***XL*** and Kawashima, arXiv:2311.05891

***XL*** and Kawashima, arXiv:2412.13758

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Global Science Graduate Course

東京大学 物性研究所

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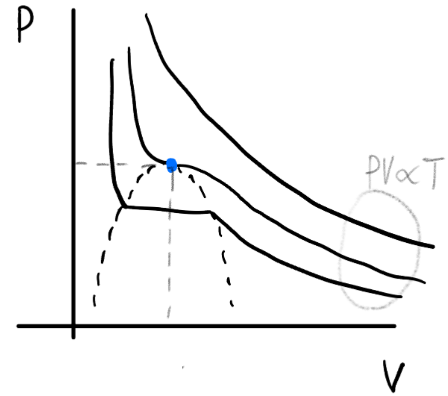
# Universality, Criticality, and RG

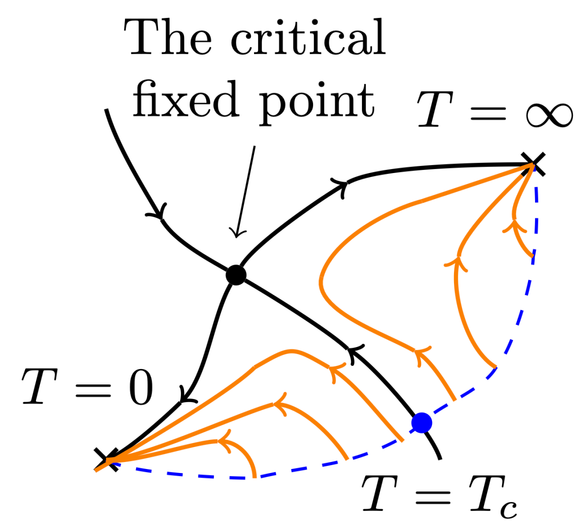
$G$
Einstein's GR



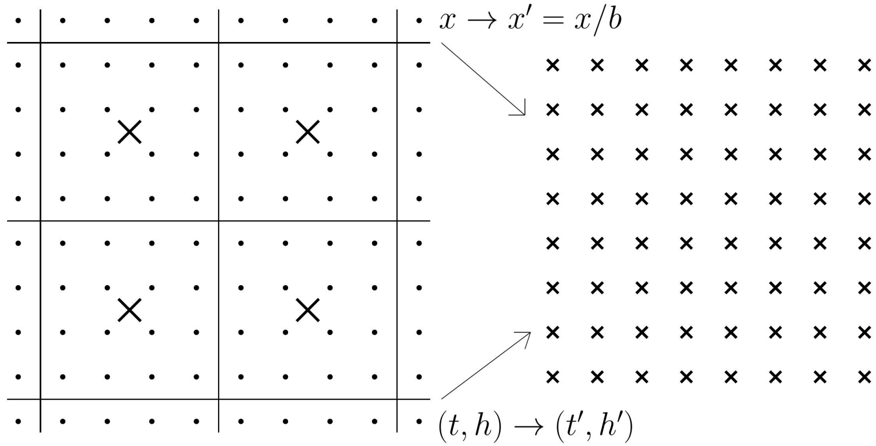

Ideal gas
temperature

When liquid-gas transition kicks in, $P_c, V_c$ depends on gas molecules.
$(P - P_c) \propto (T - T_c)^{\delta}$ → Critical exponents are universal

Due to interaction, theoretically predicting δ is challenging.
In 1960s and 70s, people like Kadanoff, Wilson, Fisher developed an idea called renormalization group (RG) to calculate these exponents.



# Block-spin: prototype of real-space RG




Wilson (1975) implemented a numerical 3x3 block-spin map by keeping 217 couplings of 2D Ising:

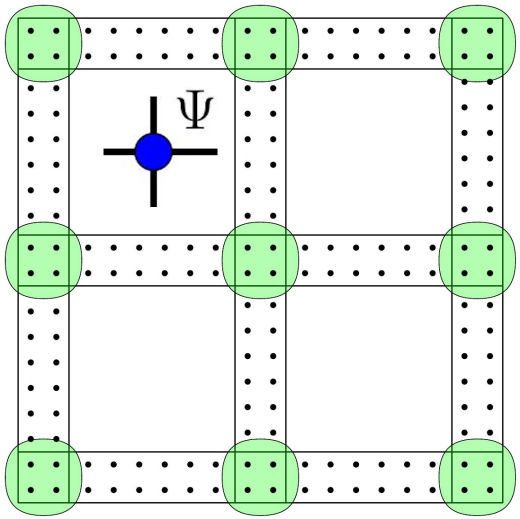
- High accuracy—1% or even 0.1% for first two exponents

"Difficult for 3D Ising… since 3x3x3 block contains about 30 spins, corresponding to $10^9$ configurations"

Migdal-Kadanoff bond moving (1976) gives $x_\epsilon = 2.1$ (best-known value is 1.41) for 3D Ising; the relative error is about 50%…

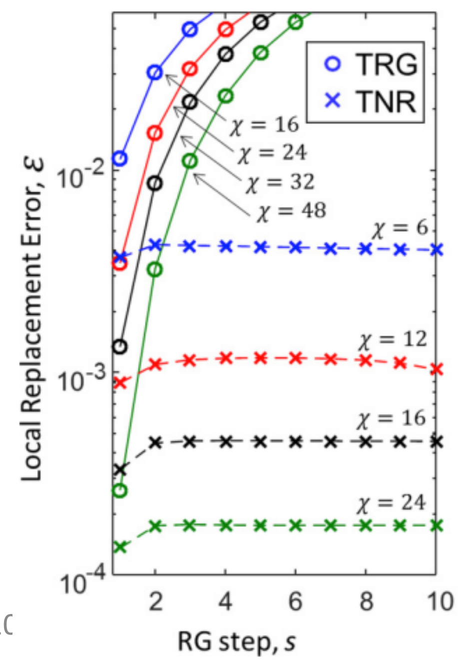
- Uncontrolled approximation
- One-shot approximation

# Tensor-network reformulation




2D classical → 1D quantum chain (radial quantization) → Entanglement-entropy area law: $S(L) \sim S_0$ [due to Levin and Nave, *PRL* **99**, 120601 (2007)]

Constant $S_0$ can justify the practice of keeping constant number of couplings!




*Systematically improvable 2D real-space RG!*

| exact | TNR(6) | TNR(16) | TNR(24) |
|---|---|---|---|
| 0.125 | 0.125679 | 0.124941 | 0.124997 |
| 1 | 1.001499 | 1.000071 | 1.000009 |
| 1.125 | 1.125552 | 1.125011 | 1.124991 |
| 1.125 | 1.127024 | 1.125201 | 1.125027 |
| max err. | 0.83% | 0.046% | 0.0069% |

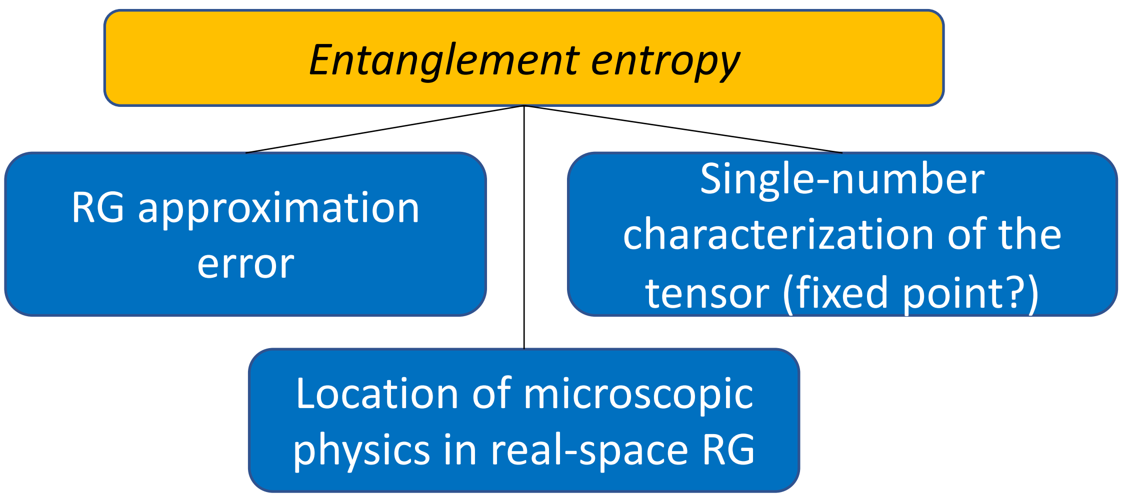
Evenbly and Vidal, *PRL* **115**, 180405 (2015)

# EE and Tensor-Network RG

Real-space RG methods often *work better in low dimensions*, but *struggle more in higher dimensions*:

- Migdal-Kadanoff bond moving can be intuitively seen as a perturbative approach starting from $d_L$
- Computationally, dimensionality of coupling constant space grows faster

For Tensor-Network RG, entanglement entropy is a tool for understanding

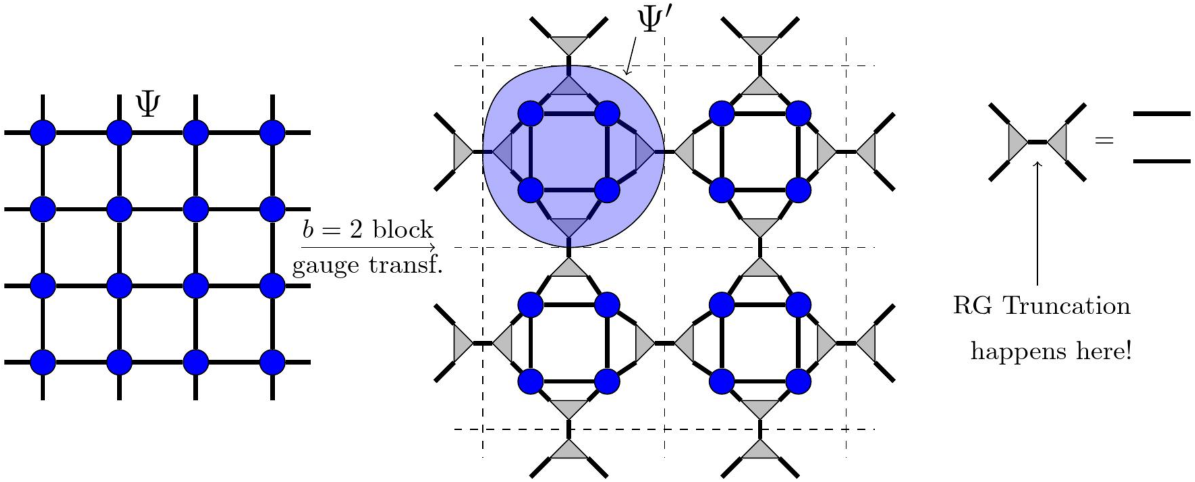


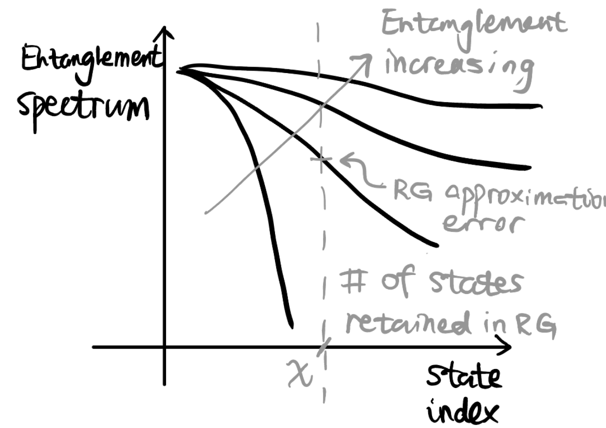



***XL** and Kawashima, arXiv:2311.05891*

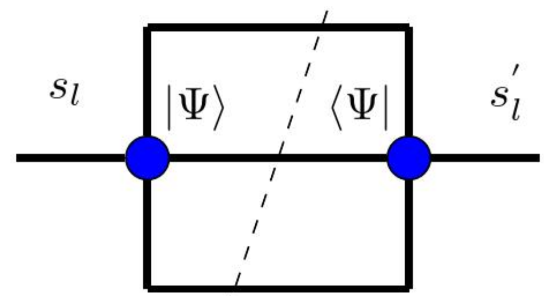
# EE and TNRG: block-tensor map

Block idea in tensor-network language: *block-tensor transformation*




An RG flow in tensor space: $\Psi^{(0)} \to \Psi^{(1)} \to \Psi^{(2)} \to \cdots$




$$\rho_{s_l, s_l'} \equiv \mathrm{Tr}_{s_r, s_u, s_d} |\Psi\rangle\langle\Psi|$$

Takeaways:

- Entanglement entropy ↗ indicates RG error ↗
- Changing entanglement entropy indicates your tensor isn't fixed (but we *wish* to have a fixed-point tensor).



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# EE and TNRG: block-tensor in 3D

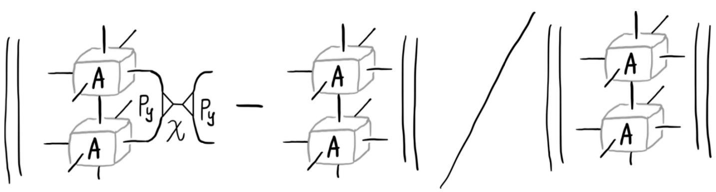
UV physics → $\alpha$; Universal physics → $F$

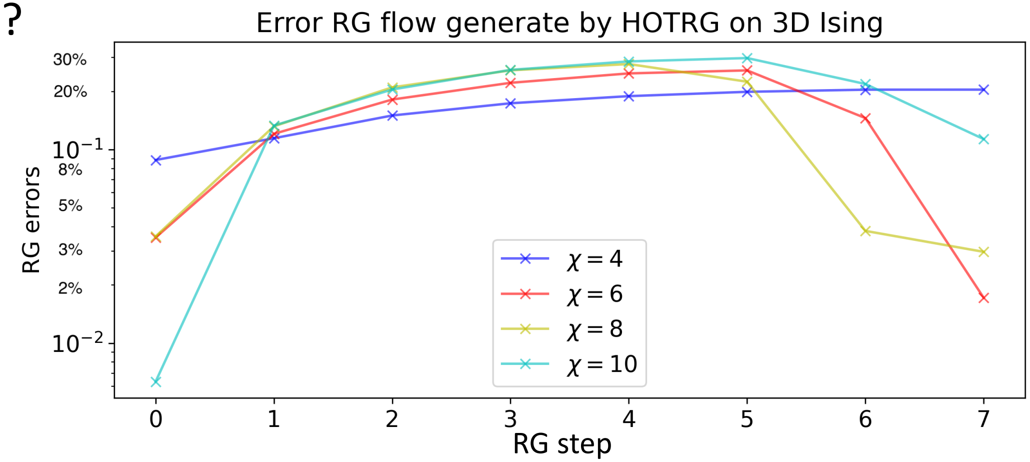
$$S(L) = \alpha L - F$$

Linear growth of $S$ marks a *qualitative* difference between 3D and 2D for block-tensor RG!

Consequences on the numerical side?

- Large RG truncation errors
- Increase states doesn't help

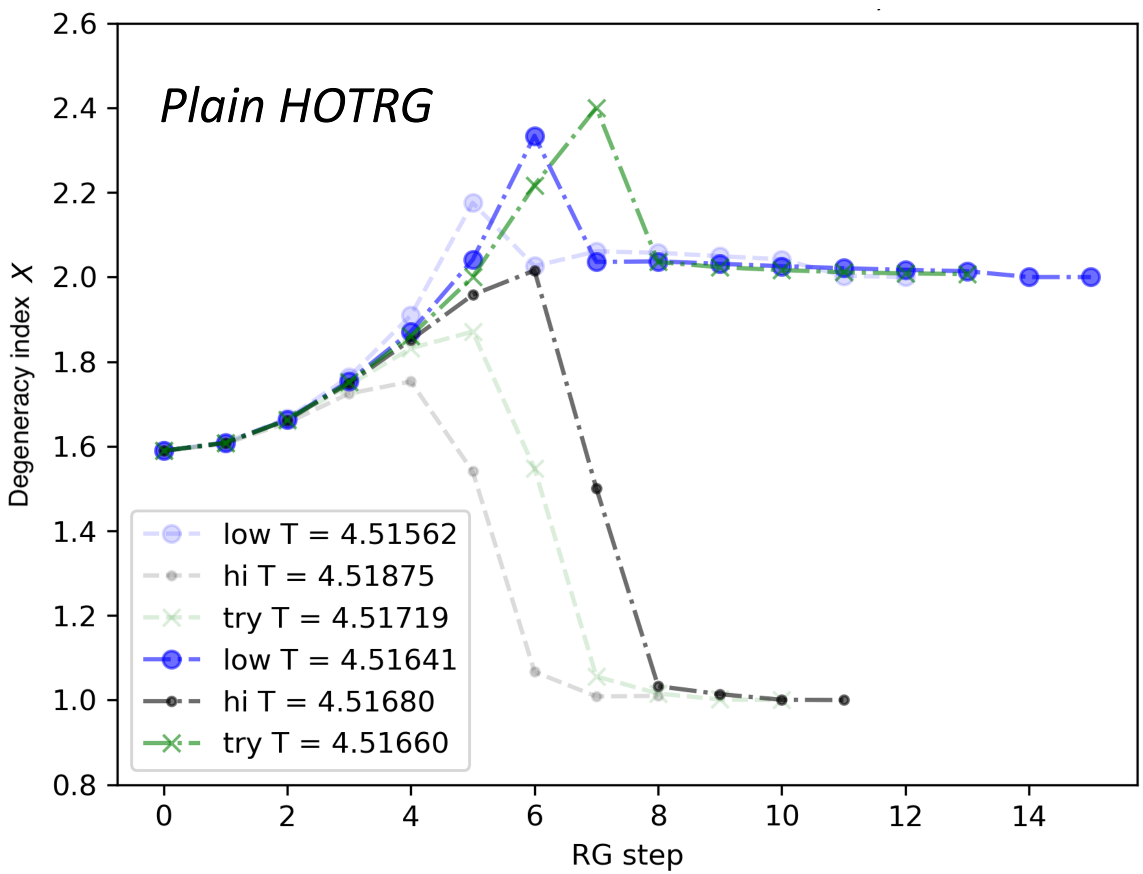


# Block-tensor transformation in 3D

*We perform a thorough analysis for bond dimensions up to 20*

Estimates fail to convergence w.r.t RG step!



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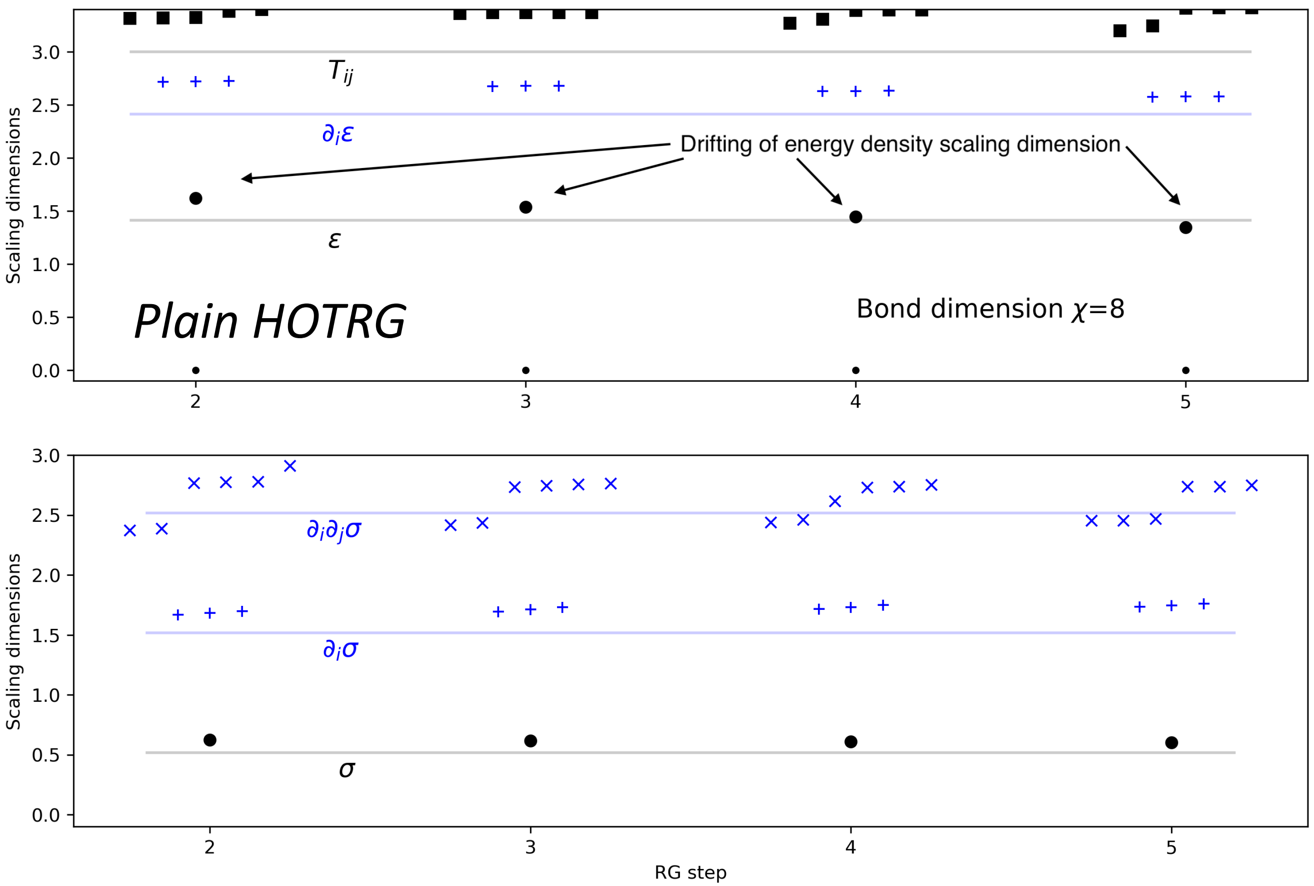
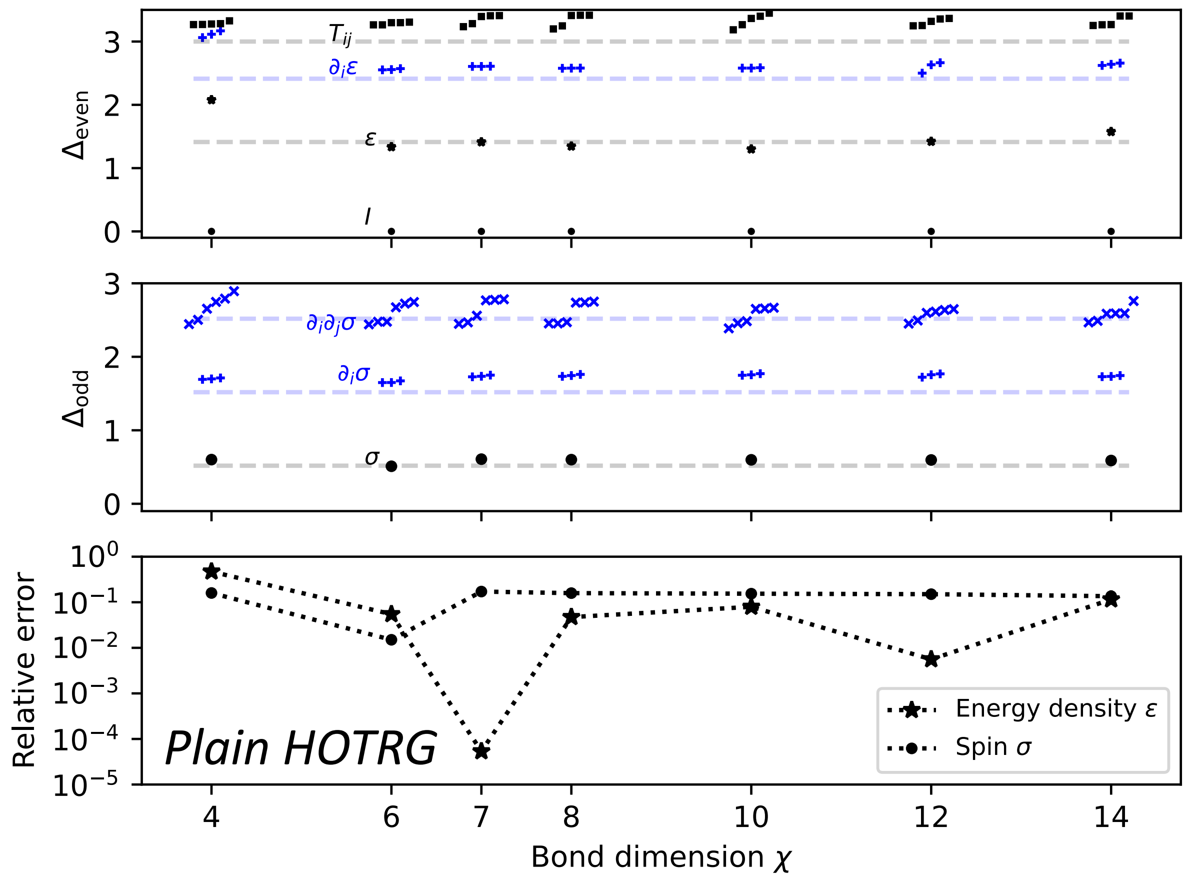


# Block-tensor transformation in 3D

- Estimated **scaling dimensions** Δ versus **the bond dimension** $\chi$

(Choose the estimates that are closest to the known value)

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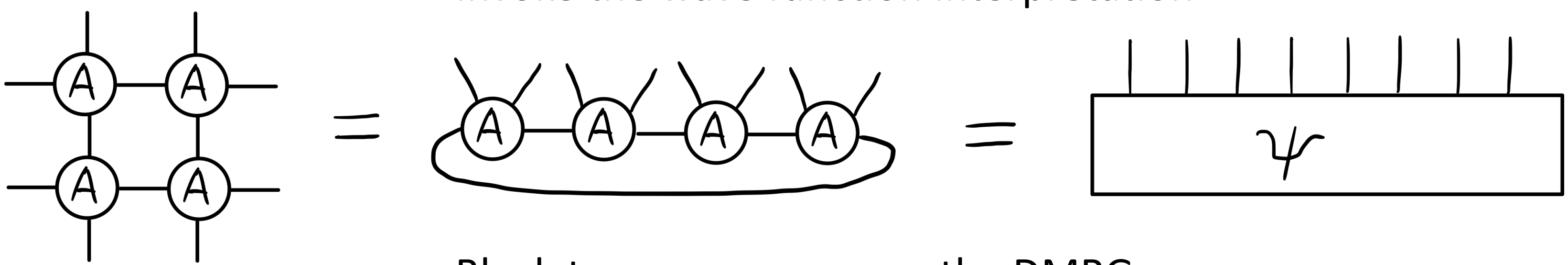
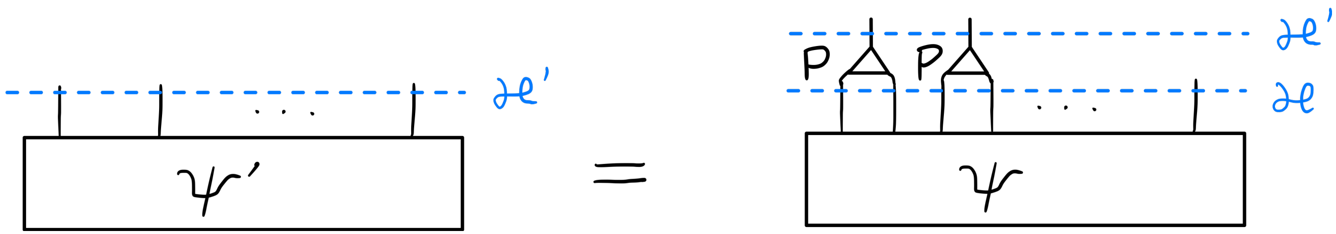
# Entanglement filtering: basic idea

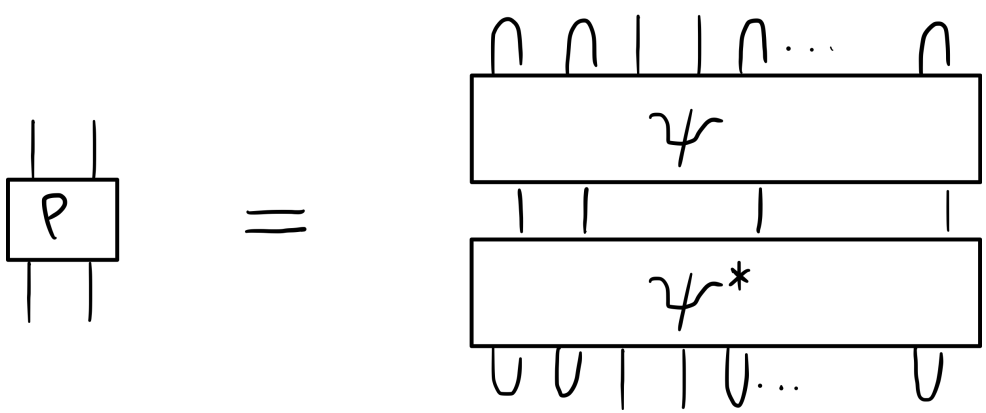
Area law can be circumvented in coarse-grained description if the boundary of the block is ”dissolved”

Invoke the wave function interpretation




Block-tensor map seen as the DMRG



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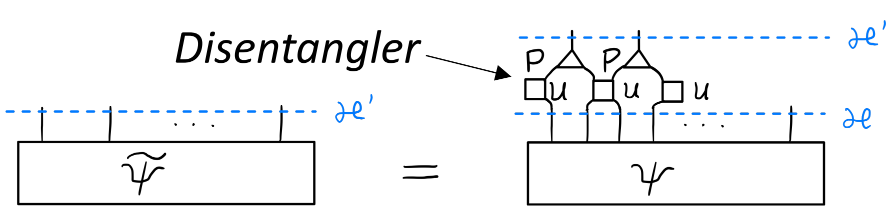
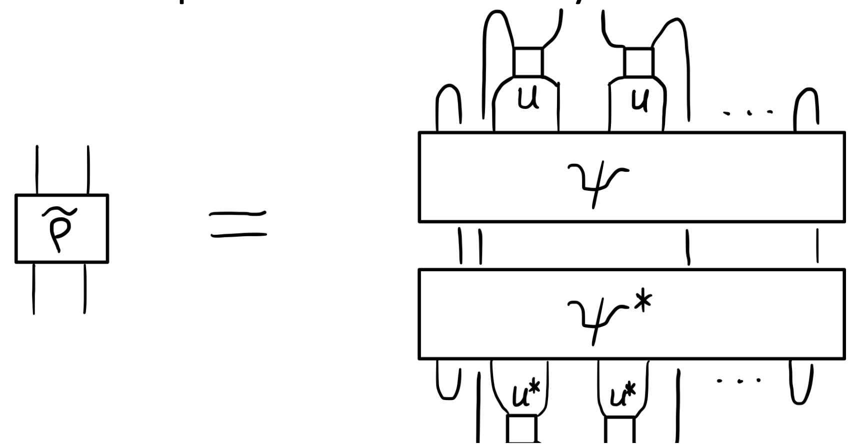
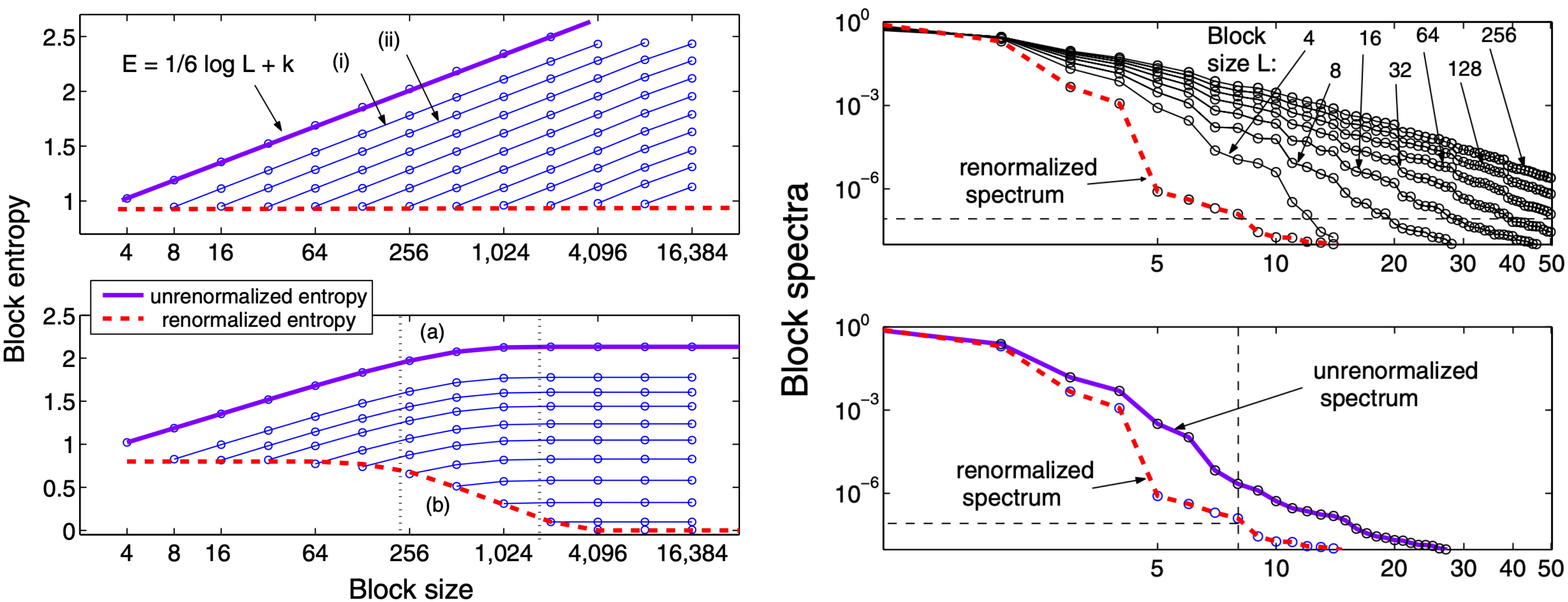


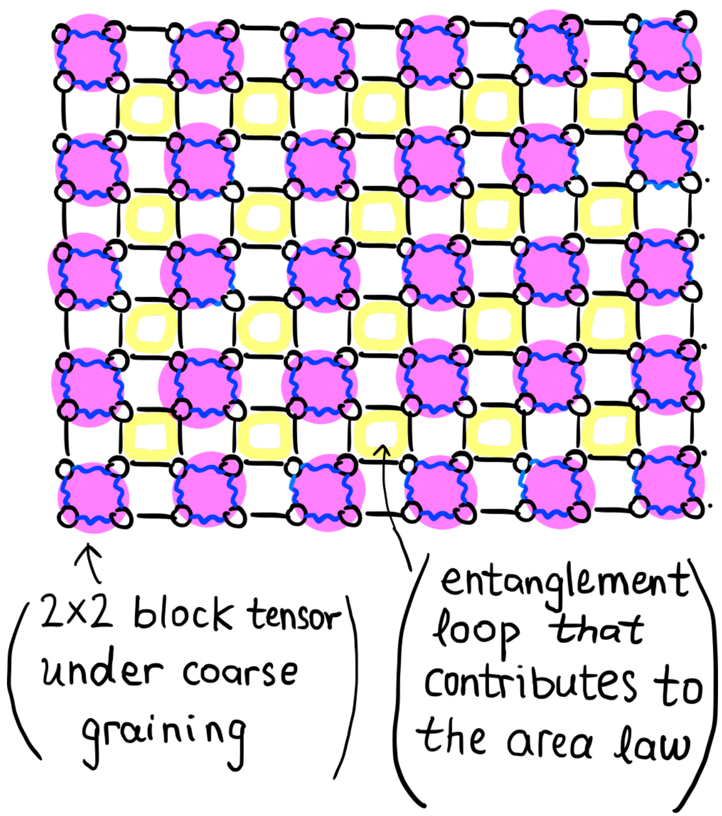




Vidal, *PRL* **99**, 220405 (2007)

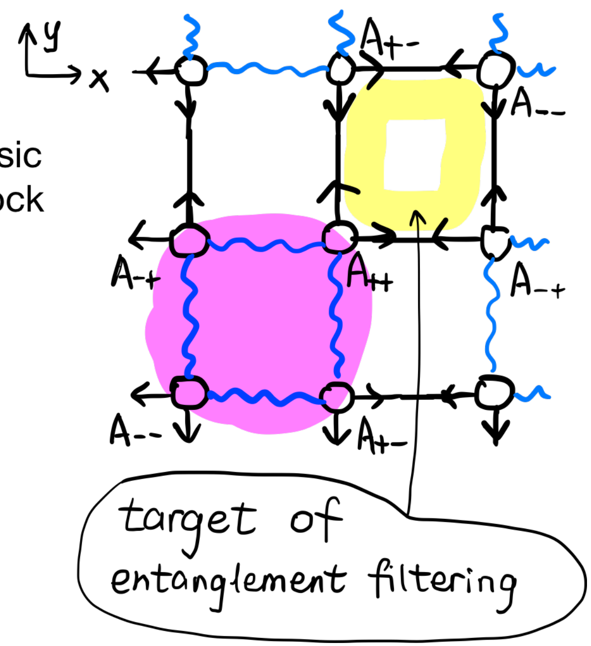


# Proposed filtering scheme

Demonstrated in the 2D square lattice, here is how to *integrate Entanglement Filtering into a block-tensor transformation*:




with the basic building block patch as

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# Proposed filtering scheme

We adopt the graph-independence idea in GILT

+

Use another way to find the filtering matrix: full environment truncation

Hauru, Delcamp, and Mizera, *PRB* **97**, 045111 (2018)

Evenbly, *PRB* **98**, 085155 (2018)

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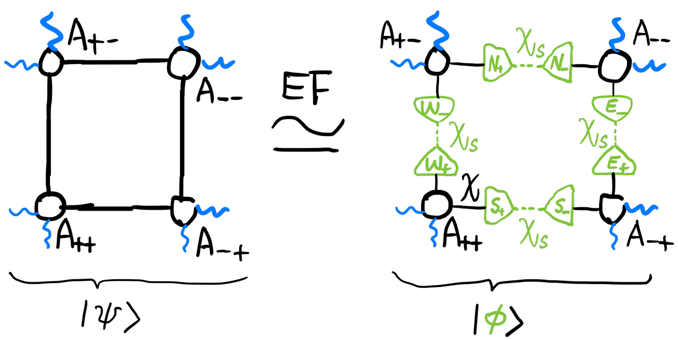
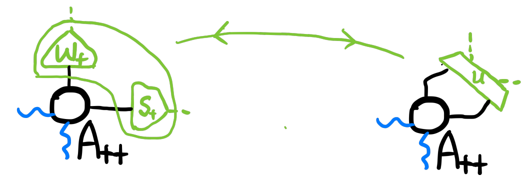
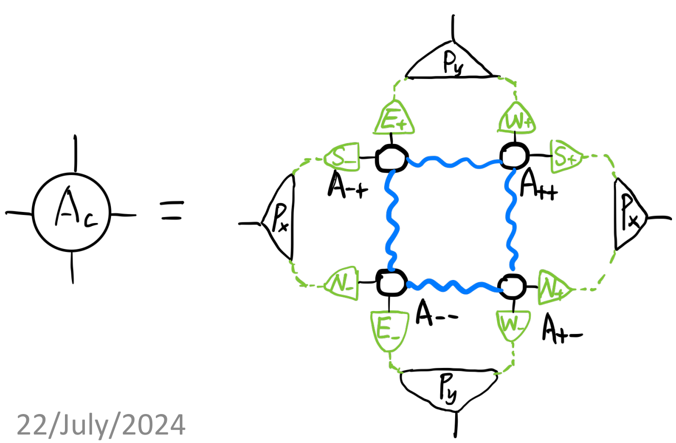
Demonstrated in the 2D square lattice, we propose:

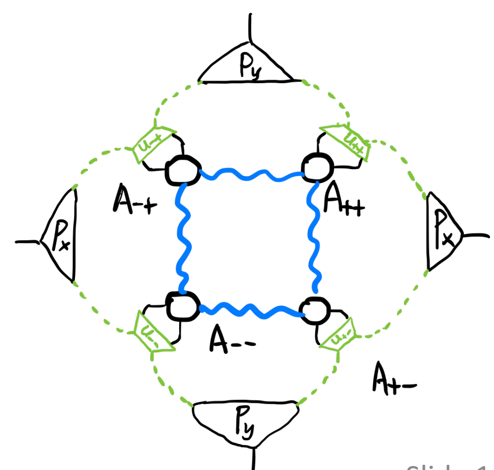



Disentangler interpretation:




Disentangler interpretation:



# Entanglement filtering in 3D

Entanglement entropy grows in 3D:

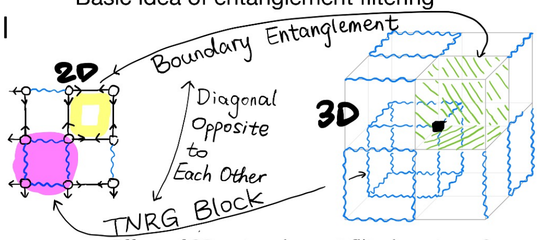
$$S = \alpha L - F$$

*Fixed # of couplings:*

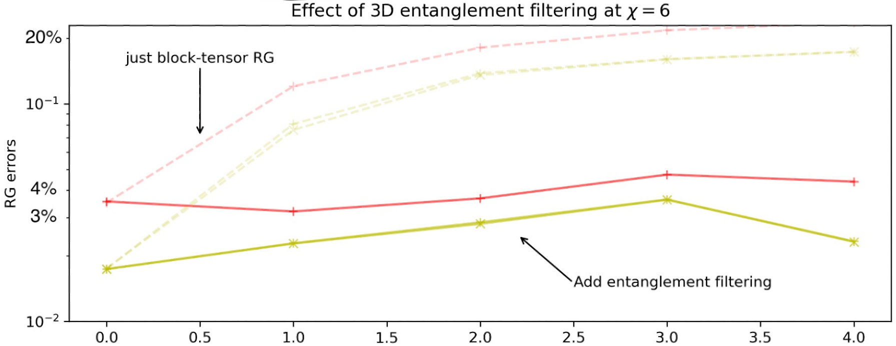
*Filtering out the boundary entanglement is essential in 3D!*

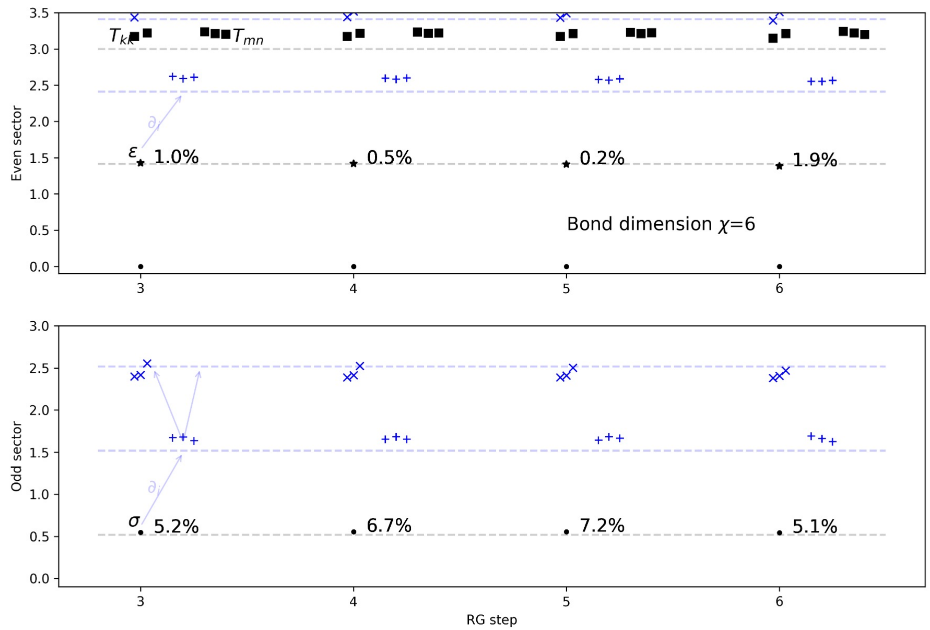
2D filtering:
Evenbly and Vidal
Hauru *et al.*

Basic idea of entanglement filtering




Effect of 3D entanglement filtering at $\chi = 6$




Note: the accuracy of exponents $x_\epsilon, x_\sigma$ ranges from 1% to 0.01% for the majority of well-developed methods

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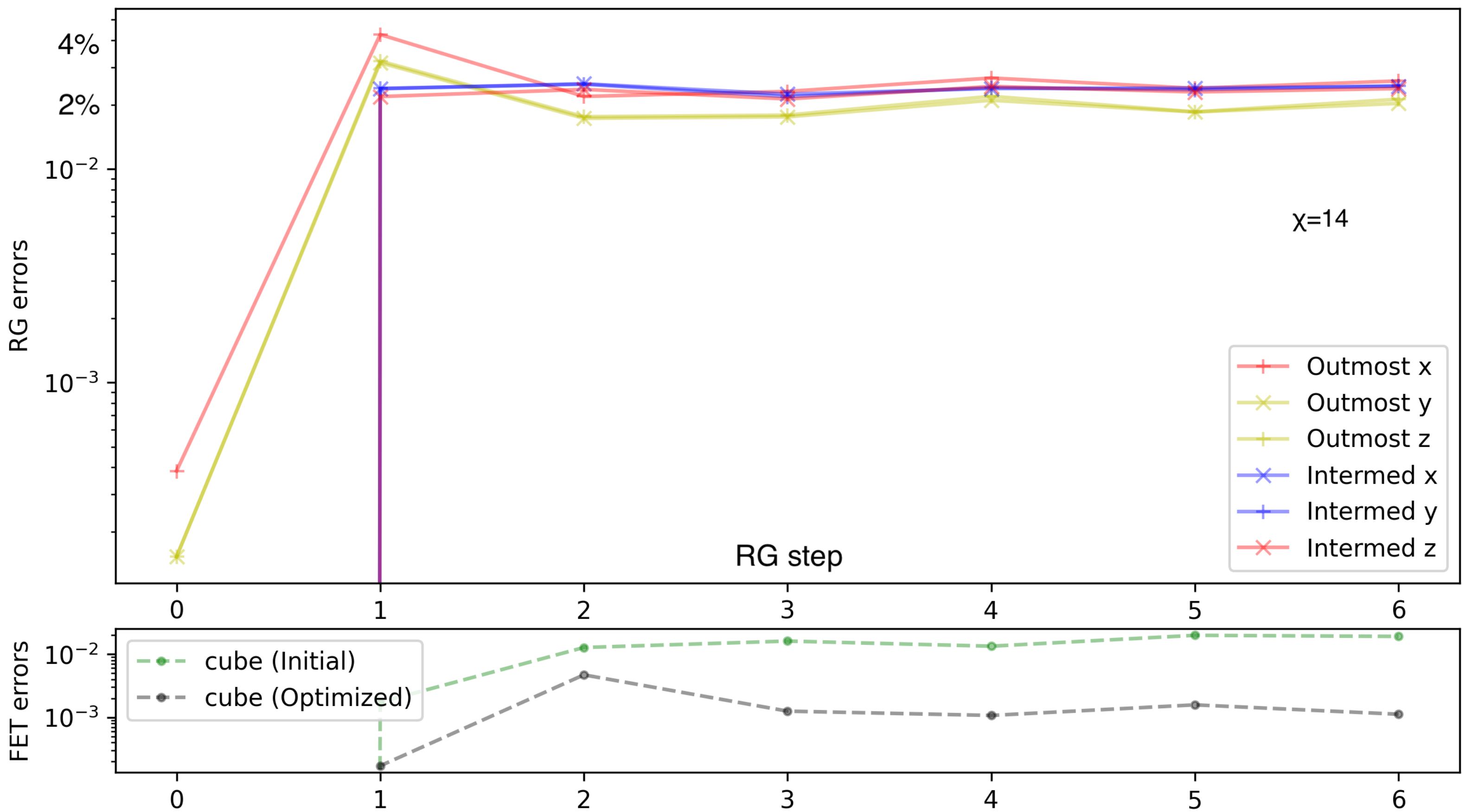
# Entanglement filtering in 3D

**RG truncation errors** versus the **bond dimension** $\chi$

| $\chi$ | 6 | 8 | 11 | 14 |
|---|---|---|---|---|
| RG error | 6% | 7% | 4% | 2% |

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# Entanglement filtering in 3D

**Scaling dimensions** versus the **bond dimension** $\chi$

| $\chi$ | 6 | 8 | 11 | 14 |
|---|---|---|---|---|
| min error | 5% | 4% | 3% | 0.4% |
| max error | 8% | 6% | 6% | 0.5% |

Table 8.1: Estimation errors for $x_\sigma$ versus bond dimension

| $\chi$ | 6 | 8 | 11 | 14 |
|---|---|---|---|---|
| min error | 0.1% | 4% | 1% | 2% |
| max error | 1% | 5% | 6% | 4% |

Table 8.2: Estimation errors for $x_\epsilon$ versus bond dimension

For spin field $x_\sigma$

- ✓ Mild decay of error with increasing bond dimension
- ✓ The magic bond dimension is $\chi = 14$

For energy density field $x_\epsilon$

- ✓ Decay of error isn't clear; but there is no apparent increase either.
- ✓ The magic bond dimension is $\chi = 6$

*Remark: in 2D TNR, the systematical improvement is demonstrated by increasing the bond dimension* $\chi = 6 \to 16 \to 24$

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# Summary

***XL*** and Kawashima, arXiv:2311.05891

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- The Kadanoff's block idea has been upgraded to become a *reliable* 3D real space RG
- In its best scenario, the error of $x_\sigma$ is 0.4% and that of $x_\epsilon$ is 0.1%

$x_\sigma, x_\epsilon$ $\qquad$ $m \sim (\lambda - \lambda_c)^\beta$

| TN Methods | Proposed | HOTRG | 2D MERA | iPEPS |
|---|---|---|---|---|
| Smallest error | 0.1%, 0.4% | 0.9% | 1.0% | 1.7% |
| Computional cost | $\chi^{12.5}$ | $\chi^{11}$ | $\chi^{16}$ | $D^{10\sim14}$ |

- The *conformal tower structure* is unique among all well-established numerical techniques
- It is a solid step towards a systematically improvable numerical RG