Tensor Network 2025（2025.07.16-18, Tsukuba U, Tokyo campus, Myogadani）

Real Space Renormalization Group with Tensor Networks

東京大学
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Collaborators

Kenji HOMMA (ISSP) Nuclear norm regularization

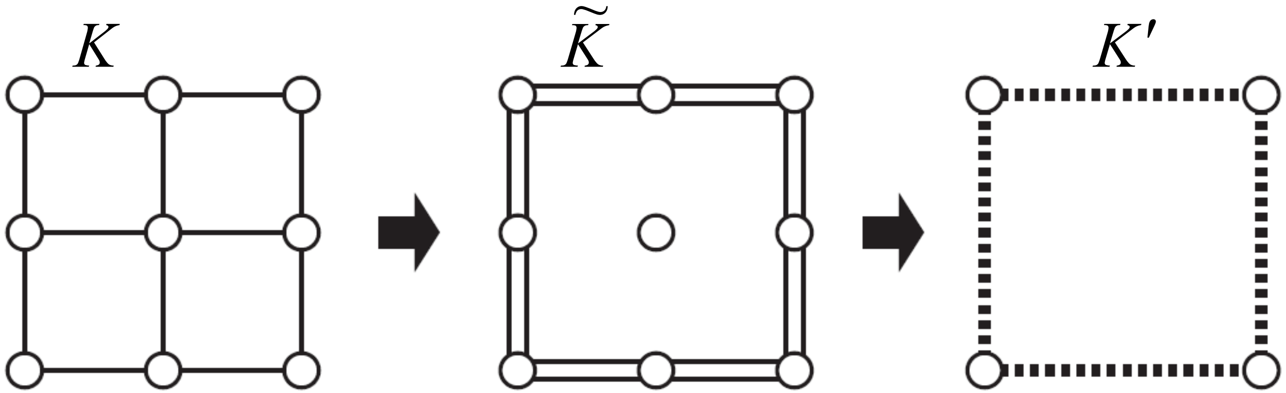
Xinliang LYU (IHES, France) Linearized TNRG, Entanglement filtering for 3D

Satoshi MORITA (Keio U.) Impurity method

Tsuyoshi OKUBO (U Tokyo) Nuclear norm regularization

Migdal Kadanoff mapping

Migdal-Kadanoff approximate RG

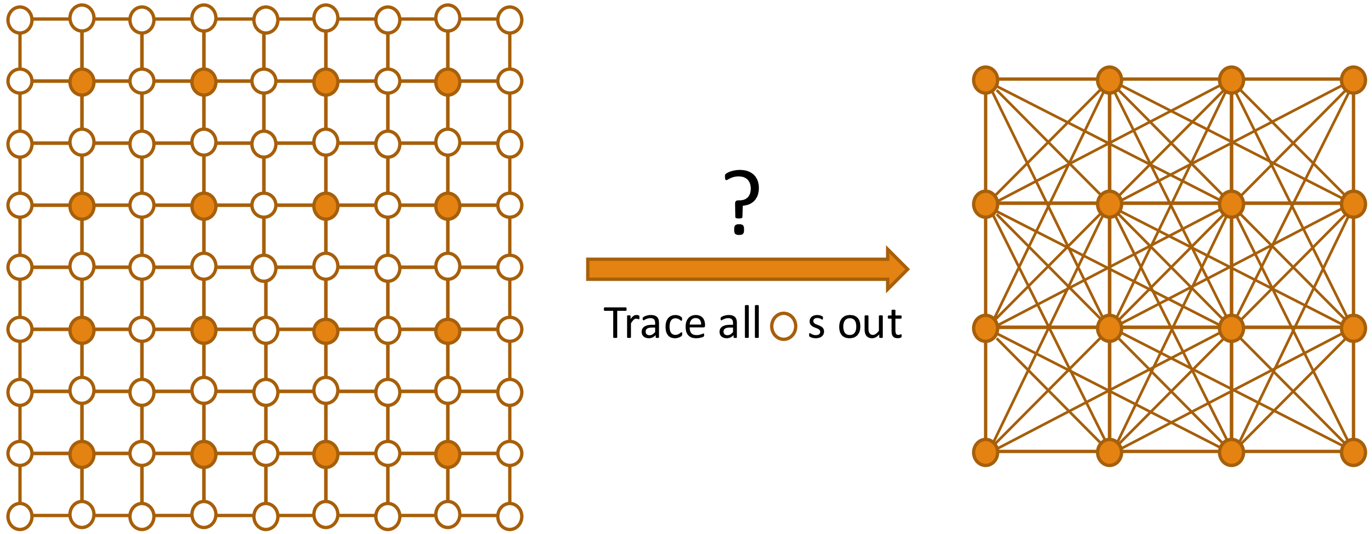


$$\widetilde{K} = 2K, \quad \tanh K' = \left(\tanh \widetilde{K}\right)^2$$
$$K' = \tanh^{-1}\left(\left(\tanh 2K\right)^2\right)$$

Simple, but not very accurate or controllable.

Even more fundamental problem in Hamiltonian formulation

We (or some of us, incl. me) usually say in lectures that an RG transformation produces a lot of compliacated interactions. **But is that all?**



We can NOT determine the state of a cluster by specifying the surrounding spins!

van Enter, Fernandez and Sokal, JSP72, 879 (1993)

In the renormalized Hamiltonian with **interactions decaying exponentially as a function of distance?**

NO!

Fundamental Problem in Hamiltonian Formulation

van Enter, Fernandez and Sokal, JSP72, 879 (1993)

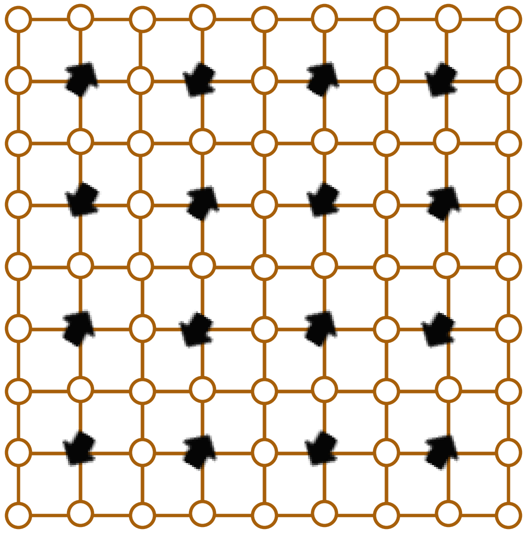
Why not?

--- Because there are some spin configurations after RG that allows the decimated spins undergoes a 1st order phase transition.

The ○ spins form a spin system on the decorated square lattice.

For the configuration of spins in the right figure, they have no effects on ○ spins because of cancellation.

Therefore, below the Tc of the decorated square Ising model, the effect of the boundary (infinitely far away) controles deep inside.



It means that the local state around the central part of this system can NOT be fixed by a finite number of spins in its neighborhood.

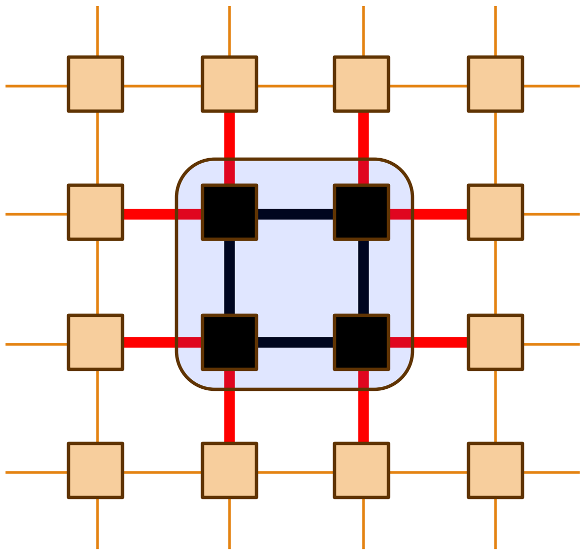
No regular Hamiltonian can produce such a property

Markov Property of TN

In contrast to the Hamiltonian RG, in TNRGs, the correlation between inside and outside of a cluster can flow only through the boundary indices.

(By fixing the boundary indices, we can uniquely determine the state inside the cluster, independent of the state of any other part of the environment.)

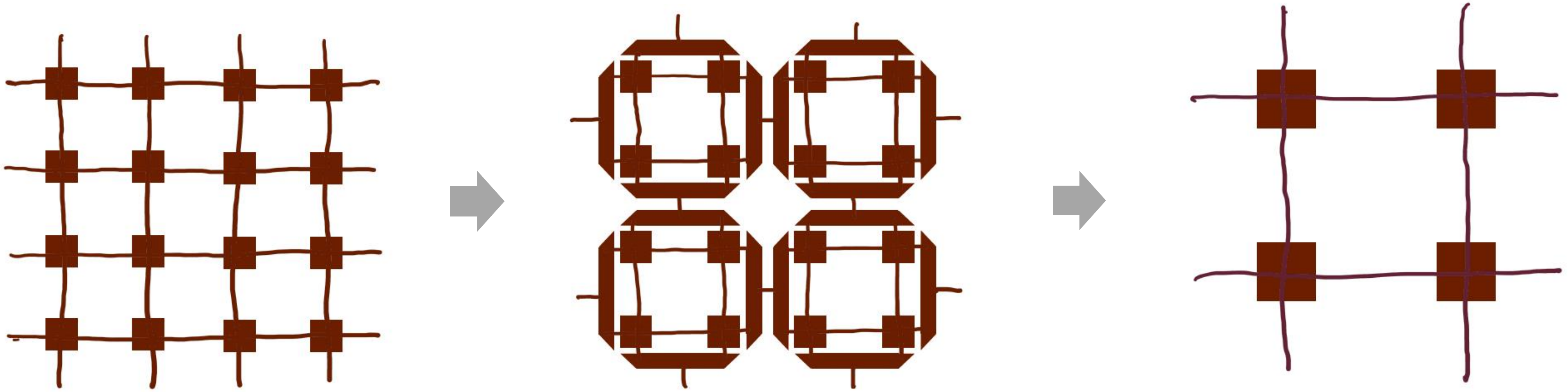
The RSRG can be well defined in terms of TN!

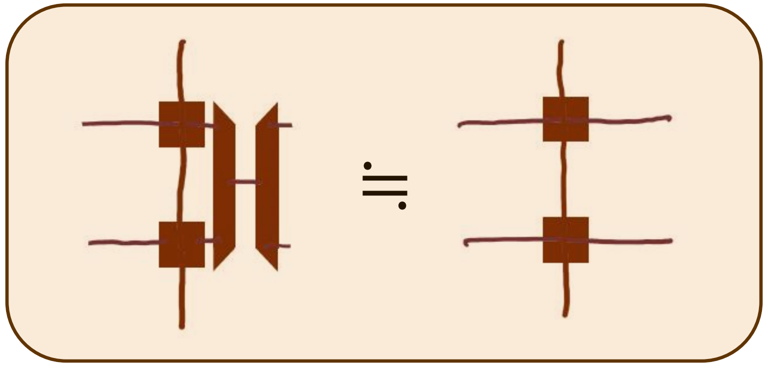


By choosing the indices on s, the state of the central cluster is uniquely determined.

HOTRG

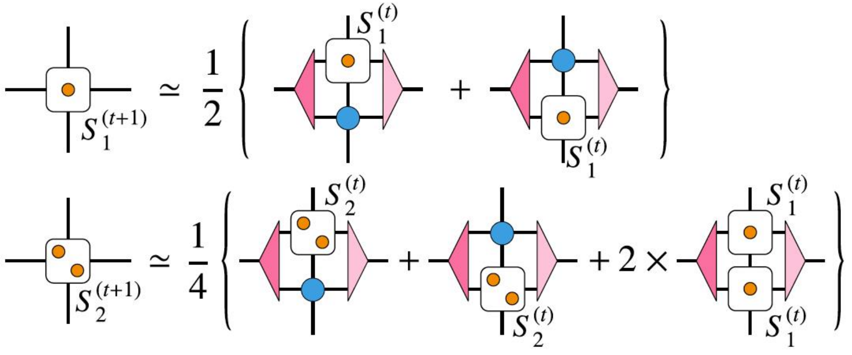
Xie, et al. Phys. Rev. B **86**, 045139(2012)



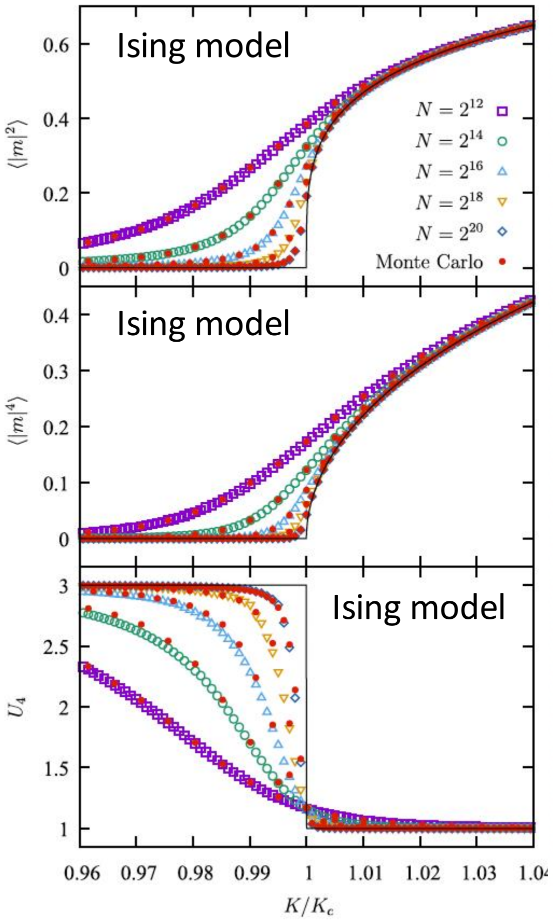


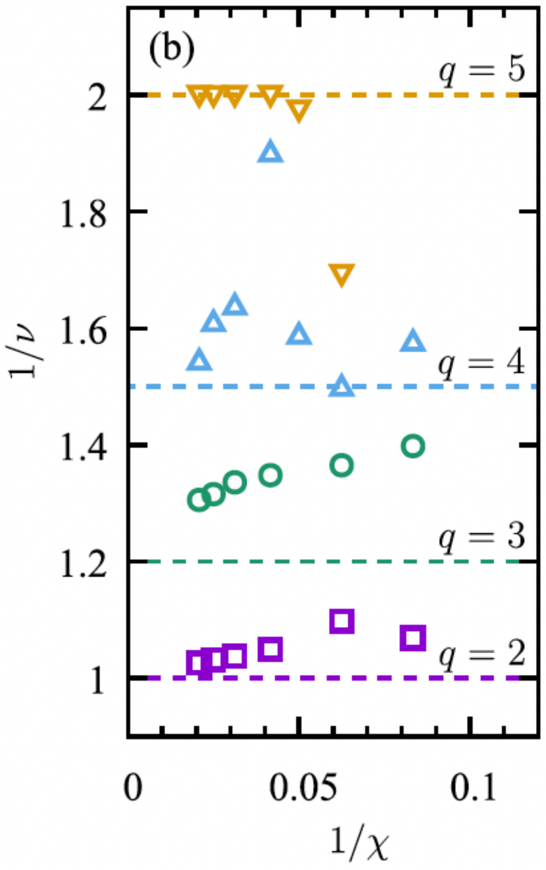
※ These diagrams are just for the concept. In Xie, et al (2012), one direction is contracted at each time.

Estimation of correlation functions
--- Impurity method ---



TNRG accurately reproduces the finite-size correction.





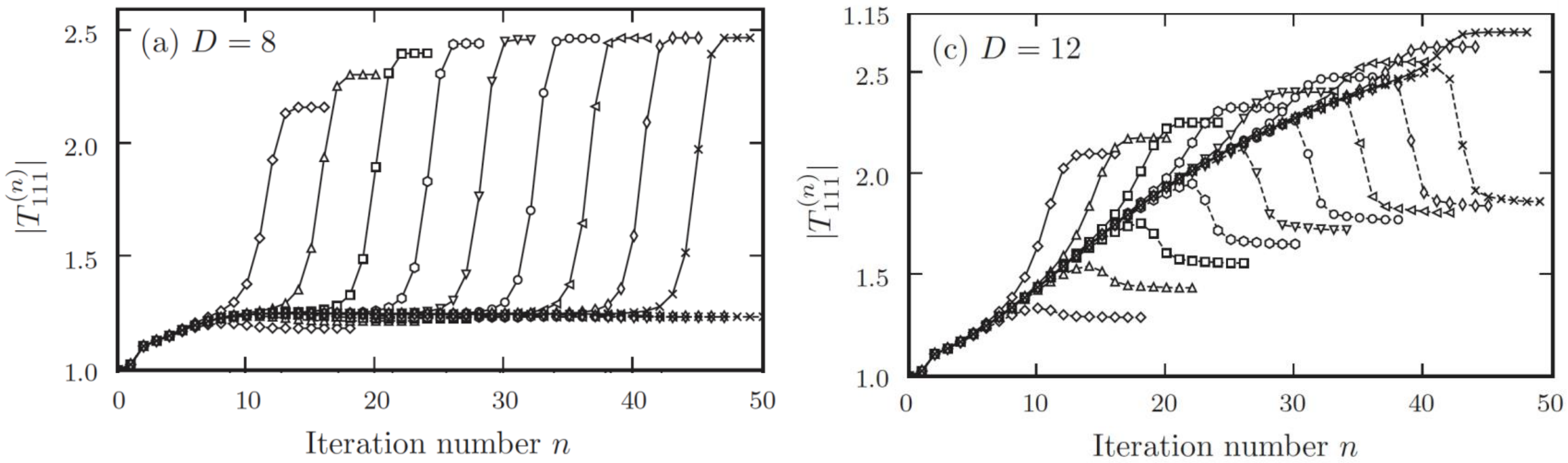
TNRG successfully identify the 1st order nature of the transition of 5-Potts in 2D.

Hinczewski and Berker: PRE 77 011104 (2008)

Convergence to fixed point

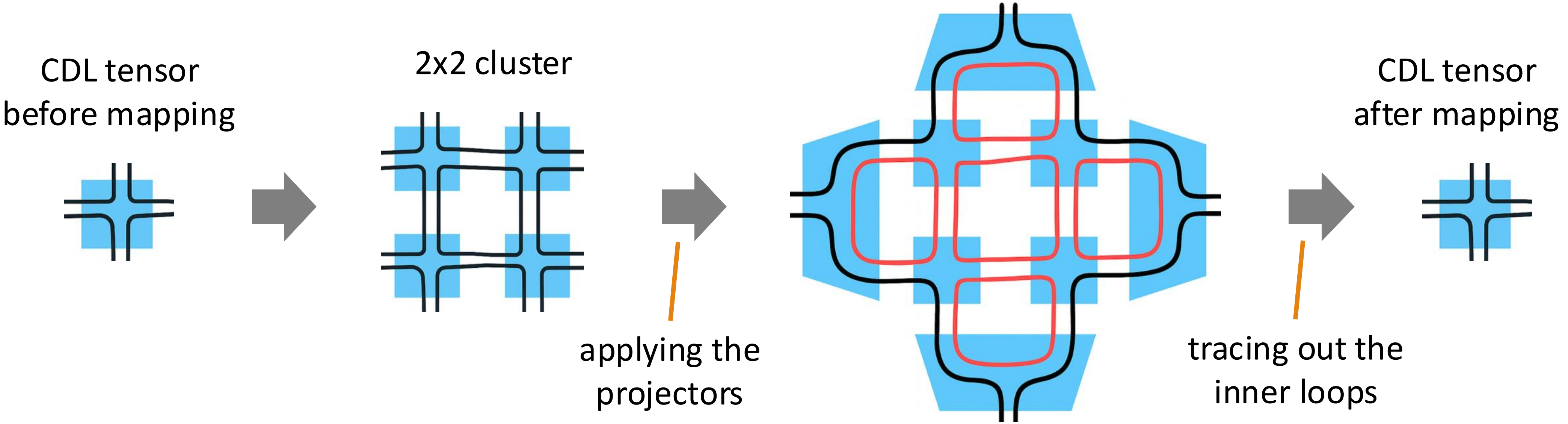
--- TRG fails in D>2, not just inaccurate ---

<u>TRG (tensor network renormalization a la Levin and Nave)</u>



We don't reach the fixed-point tensor for large bond dimensions.
(Something similar happens to HOTRG.)

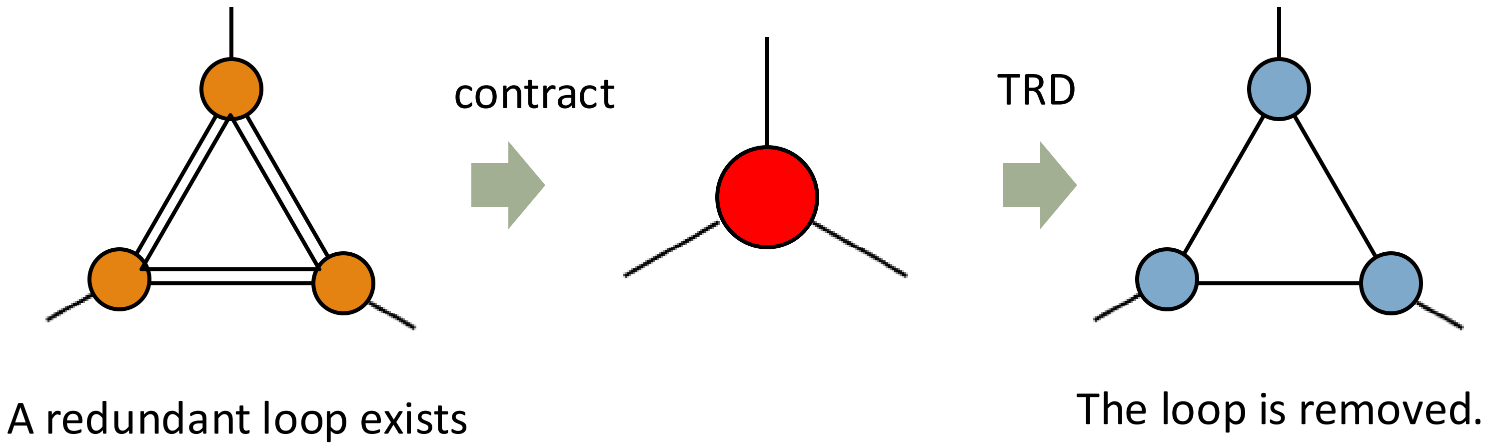
CDL tensor is a $\overbrace{\text{fixed}}^{\text{artificial}}$ point of TNRG



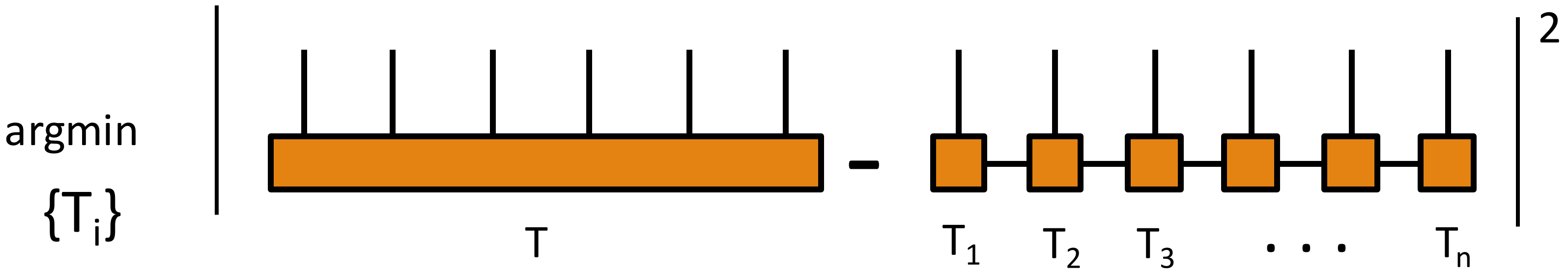
HOTRG does not change the CDL tensor.

Typically, TNRG at criticality is more complicated. Still, the CDL structure may arise, and the tensor may become a tensor product of the meaningless CDL and the core, which carries the essence of the critical properties. Once the CDL arises, it stays there and cramping the core.

If an effective tensor-ring decomposition (TRD) is available …



However, TRD is generally very hard



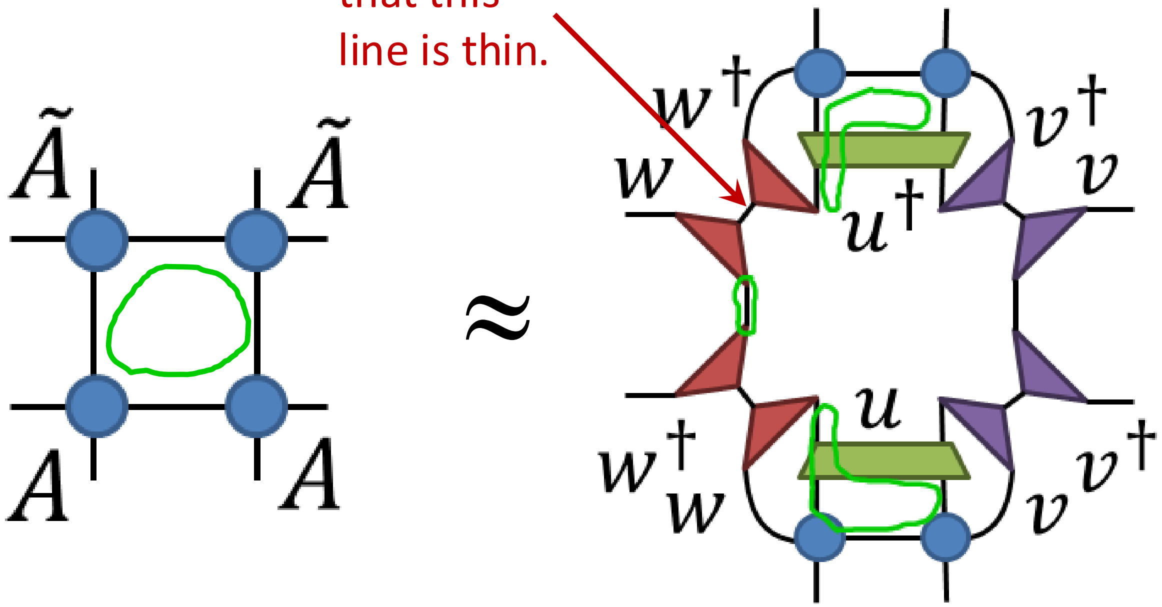
Optimal product state (rank-1 CPD) approximation is already NP-hard.
Optimal rank-2 CPD problems is undefined (even more difficult).

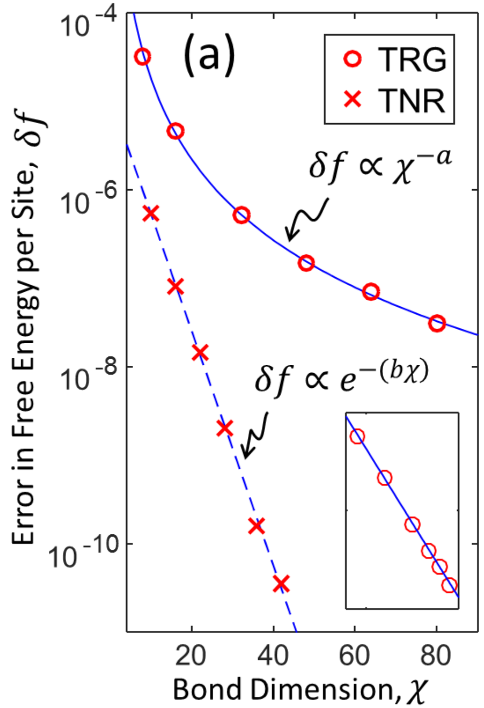
C. J. Hillar and L.-H. Lim.: In Journal of the ACM (JACM) 60.6 (2013), p. 45.
(also available as arXiv: 0911.1393)

The 1st approach

Disentangler removes loops

It is essential that this line is thin.

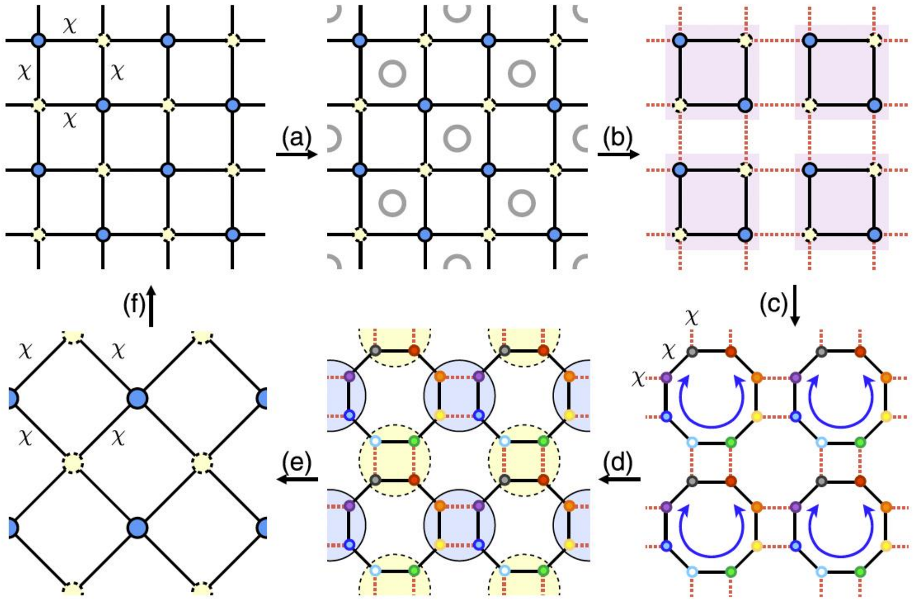




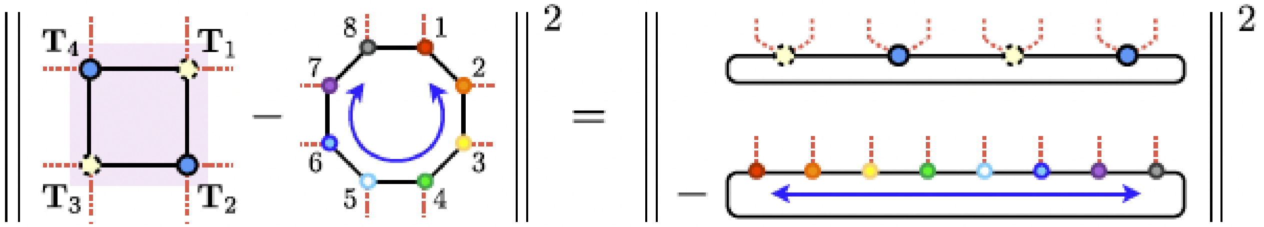
By pinching the "information path", we can split the remaining loop, and remove them at the next contraction.

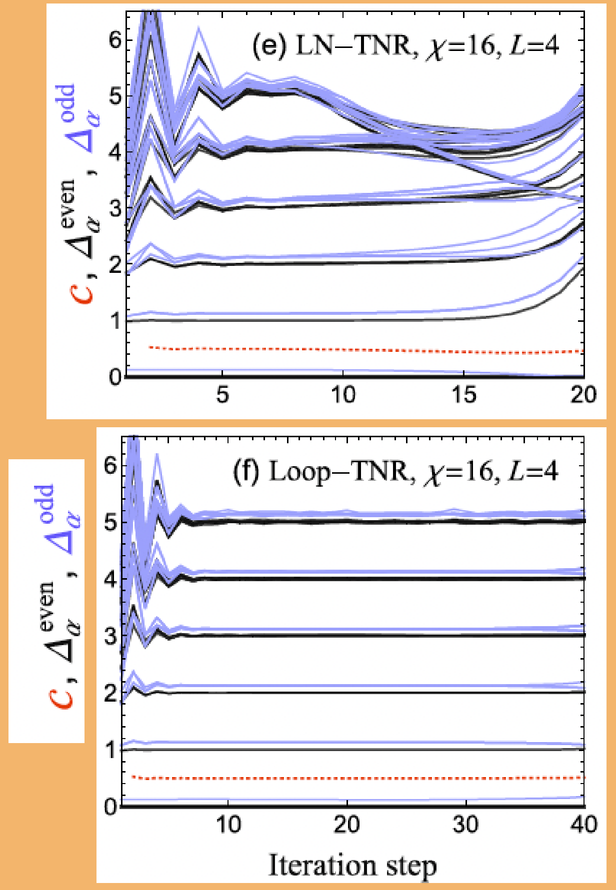
Yang, Gu, Wen: PRL 118, 110504 (2017)

Loop TNR



Loop Optimization



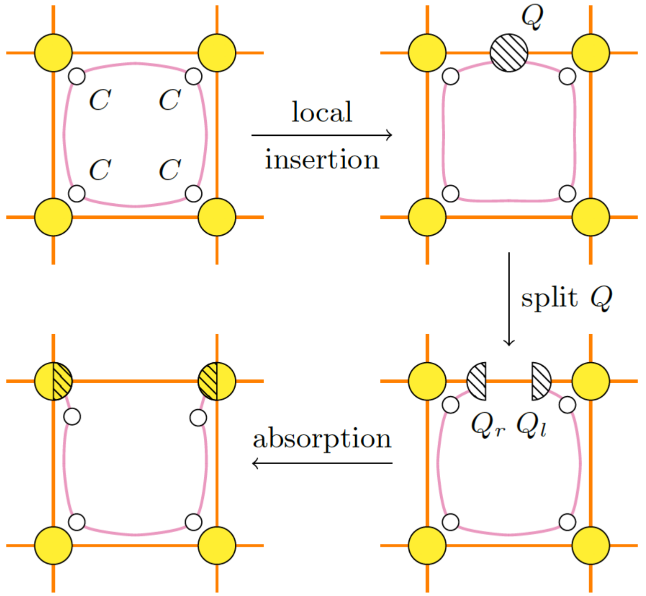


※ *The result of the TM method with L=4*

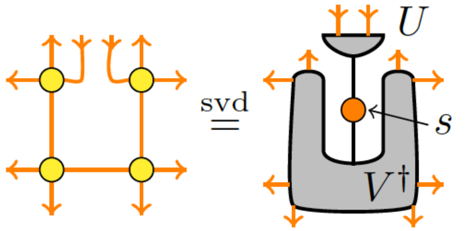
Removal of the short-range correlation

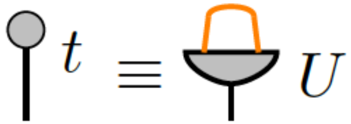
GILT

A general method for constructing a projector "Q" that cuts the redundant entanglement loops, if it exists.

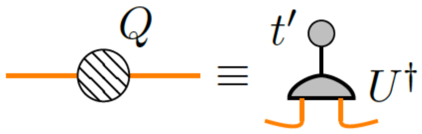


Q filters out the redundant loops of short-range correlations





$$t_i' = t_i \frac{s_i^2}{s_i^2 + \epsilon_{\mathrm{gilt}}^2}$$

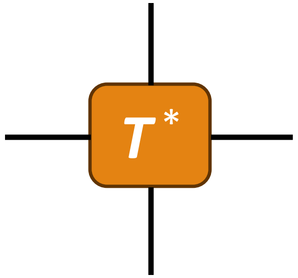


Nuclear Norm Regularization (NNR)

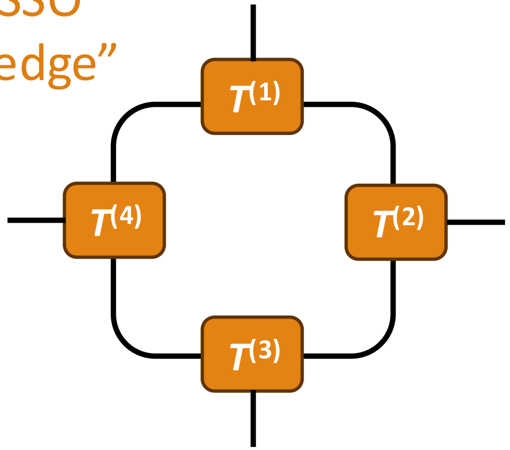
$$C(T^{(1)}, T^{(2)}, \cdots, T^{(d)}) = |\mathcal{T} - \mathcal{T}^*|^2 + \lambda \sum_{i=1}^{d} \sum_{\alpha} \left| T^{(i)}_{(\alpha)} \right|_*^{1}$$

linear norm ...
similar to LASSO
with “sharp edge”

$$(\text{Target}) : \mathcal{T}^*$$



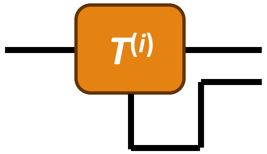
$$(\text{Ansatz}) : \mathcal{T} \equiv \text{Cont} \bigotimes_{i=1}^{d} T^{(i)}$$



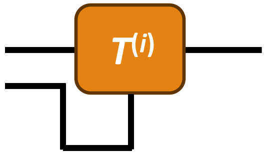
$$X_{(\alpha)} \equiv (\text{The } \alpha\text{th mode slice matrix of } X)$$

$$|M|_* \equiv \sum_{\mu} p_{\mu} \quad (p_{\mu} \equiv (\text{the } \mu\text{th sng. val. of } M))$$

$$T^{(i)}_{(1)} \equiv$$

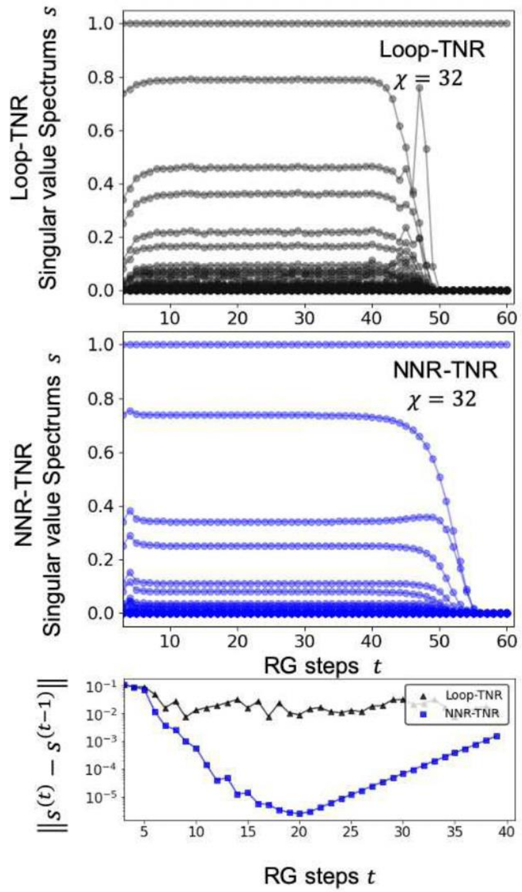


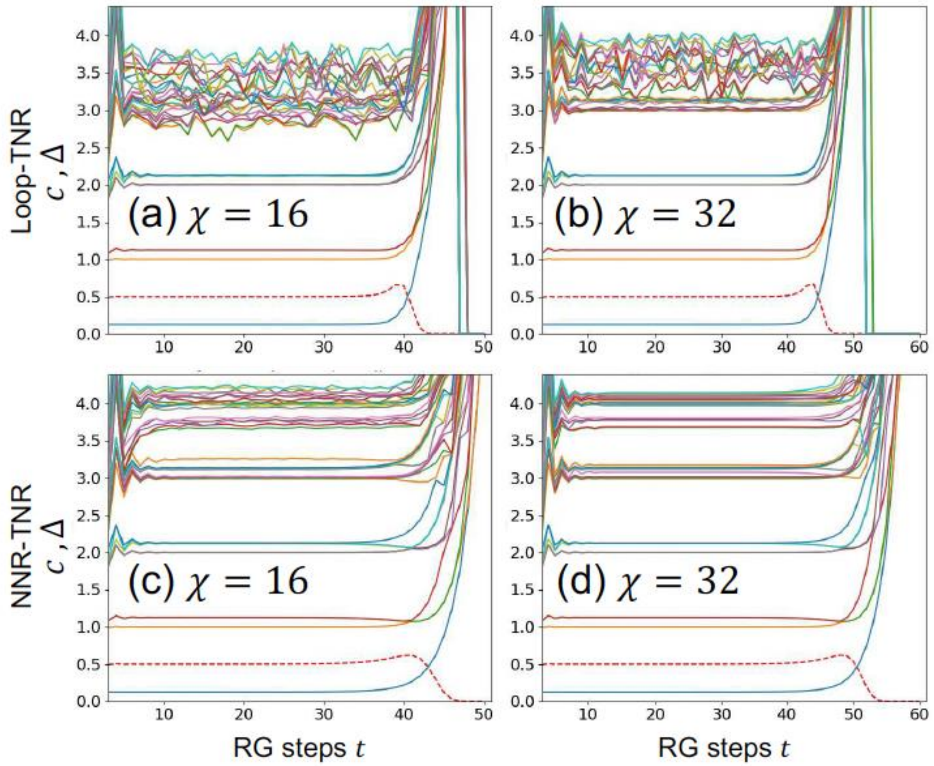
$$T^{(i)}_{(3)} \equiv$$



Very similar to entropic bias, but the bias term is LINEAR in the singular value. (CF: LASSO)

NNR-TNR (numerical stability)





“Loop TNR”

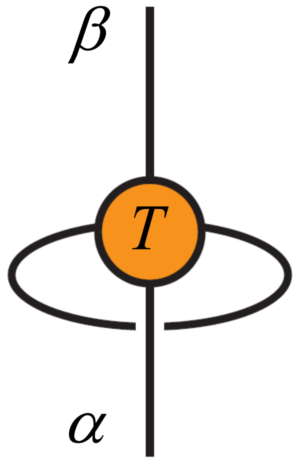
“NNR TNR”
(= Levin-Nave TRG with NNR)

※ *The result of the TM method with L=2*

NNR-TNR seems more stable than loop-TNR

Transfer matrix method

--- 2dCFT-aided method for scaling dimensions ---



$$\lambda_\mu = e^{-2\pi\left(\Delta_\mu - \frac{c}{12}\right)}$$

eigenvalues of the partially contracted scale invariant tensor

$$\zeta_s \equiv e^{-f_s} = \sum_\mu e^{-2\pi\left(\Delta_\mu - \frac{c}{12}\right)\mathrm{Im}\tau + i\sigma_\mu \mathrm{Re}\tau}$$

$$\Delta_\mu \equiv h_\mu^R + h_\mu^L \ , \quad \sigma_\mu \equiv h_\mu^R - h_\mu^L$$

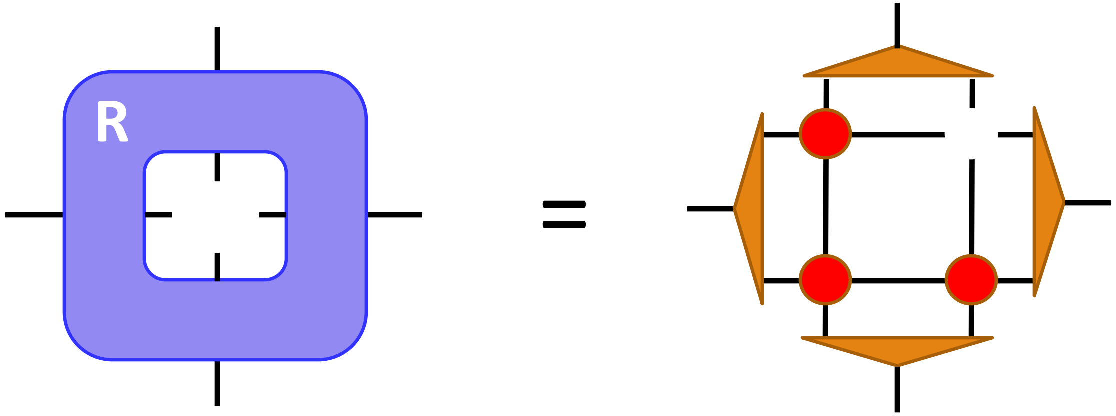
$$\tau \equiv (\text{complex aspect ratio parameter})$$

Cardy: Nucl. Phys. B 270 (1986)

It is not clear how to generalize to higher dimensions.

What about d>=3 ?
--- Textbook RG program with HOTRG ---

HOTRG ... easier to extend for higher dimensions



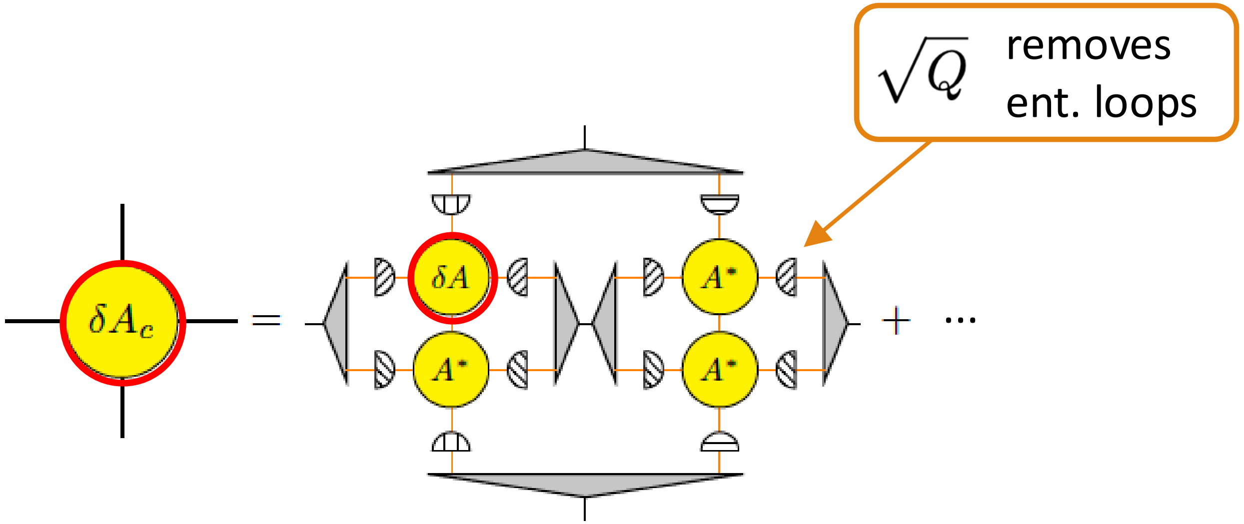
✓ Short-range correlation stays no matter how much we repeat RG steps.

Linearized Super-operator

$$A' = R(A)$$

$$\delta A' = S\,\delta A$$

$$S \equiv \frac{\partial R(A)}{\partial A}$$



In differentiating R(A) w.r.t. A, we did not take the variation in the projectors into account, i.e., the projectors are fixed.

This treatment relates S to the dilatation operator.

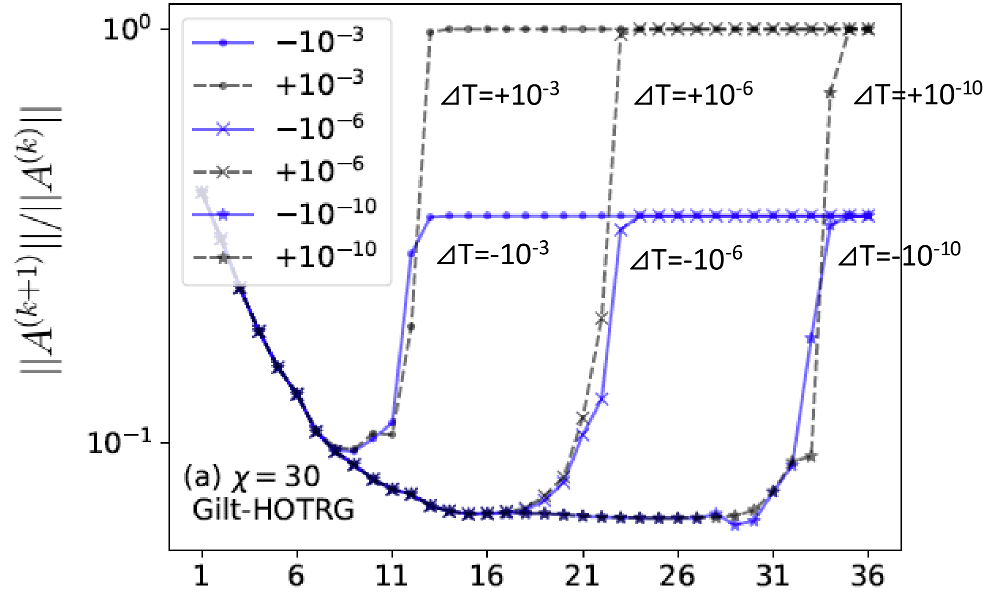
Benchmark (2d Ising)

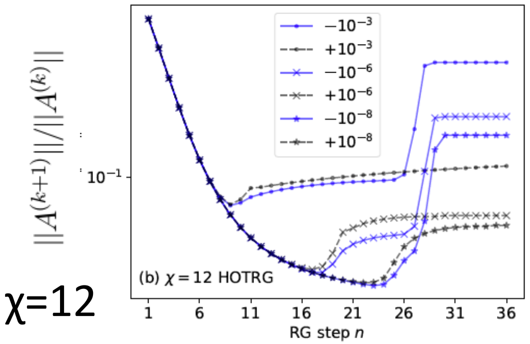
RG-step dependence of the tensor norm ratio

$$\Delta T \equiv \frac{T - T_c^{[\chi=30]}}{T_c}$$

- depending on the temperature, curves start to deviate from the critical curve.
- χ=30 stays longer at the bottom compared to χ=12, indicating approach to the true fixed point as we increase χ.

χ=30



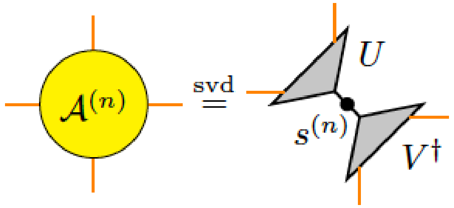


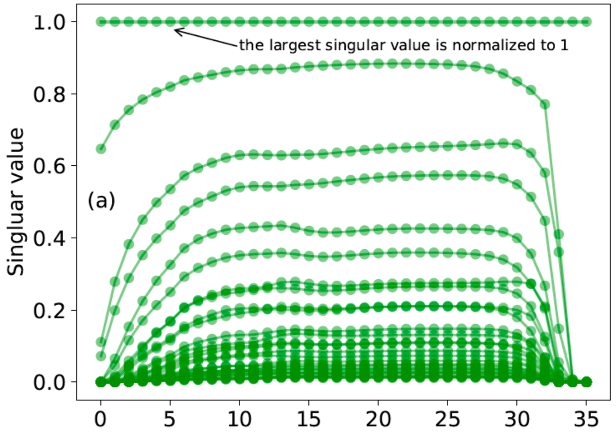
χ=12

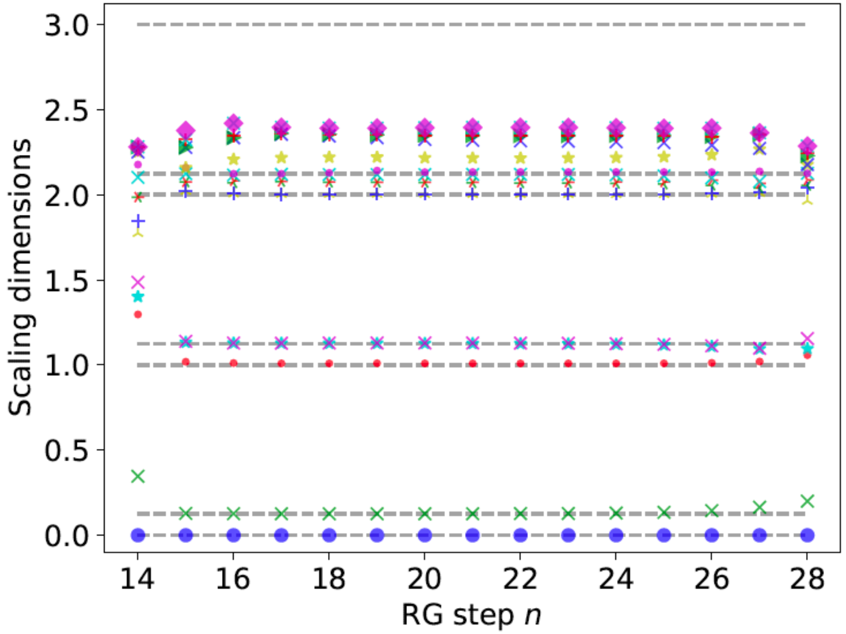
Benchmark (2d Ising)

- The method yields at least 4 digits of scaling dimensions for the most relevant ones.

Exact	0.125	1	1.125	1.125	2	2	2	2
RG pres.	0.127	1.009	1.125	1.128	2.002	2.004	2.068	2.073
Trans. mat.	0.125	1.009	1.130	1.148	1.313	1.457	1.558	1.654







Lyu, Xu and NK: PRR 3 023048 (2021)

W. Guo and T.-C. Wei, PRE 109 (2024) 034111

You can get OPE coefficient, too.

$$\left\langle T_\alpha \middle| R \middle| \begin{matrix} T_\beta & T_\gamma \\ T & T \end{matrix} \right\rangle = \frac{c^{\alpha}_{\beta\gamma}}{a^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}}$$

↑ Just the essence. In actual computation, a more sophisticated formula were used.

... The good thing is, we can use the same method for 3D or higher since we don't rely on any formula specific to 2D CFT!

	Exact	lTRG
Δ_σ	$1/8$	0.127
Δ_ε	1	1.002
$C_{\sigma\sigma\sigma}$	0	2.4×10^{-7}
$C_{\sigma\sigma\varepsilon}$	$1/2$	0.512
$C_{\varepsilon\varepsilon\sigma}$	0	1.1×10^{-6}
$C_{\varepsilon\varepsilon\varepsilon}$	0	0.168
$C_{\sigma\varepsilon\sigma}$	$1/2$	0.409
$C_{\sigma\varepsilon\varepsilon}$	0	1.5×10^{-7}

Ebel, Kennedy, Rychkov: arXiv:2408.10312

Newton method with Jacobian

$$F(T) \equiv T - R(T)$$ ← We want make this 0.

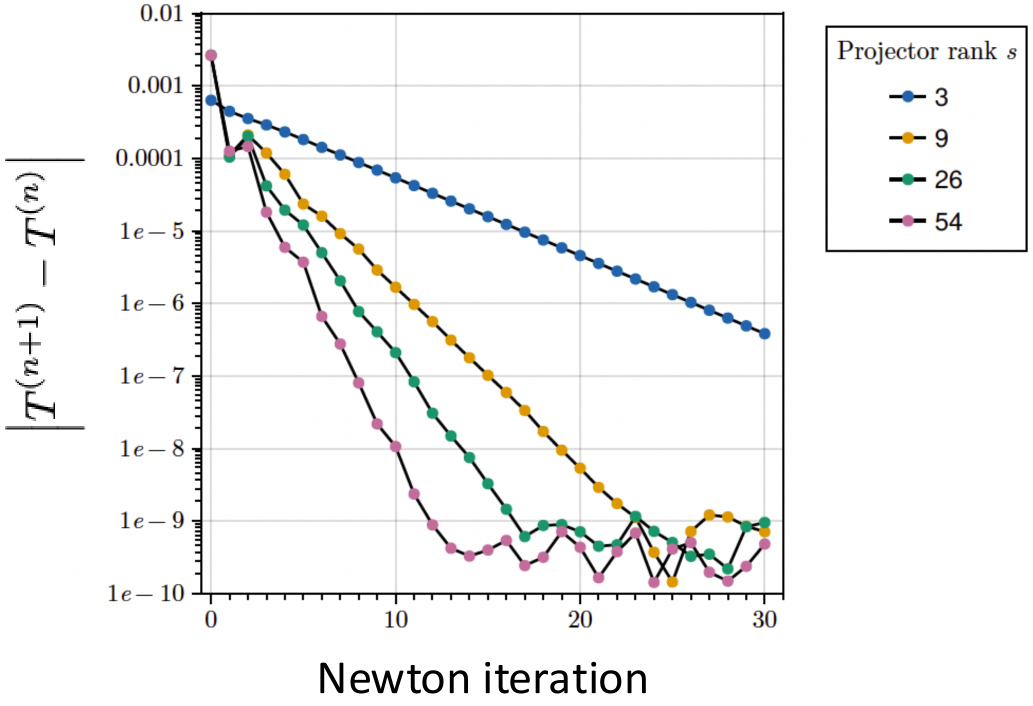
$$G(T) \equiv T - \left(\frac{dF}{dT}\right)^{-1} F$$ ← Newton map

$$T^{(n+1)} = G(T^{(n)})$$ ← Newton iteration

To make the burden lighter, the gradient is replaced as

$$\frac{dF}{dT} = I - \frac{dR}{dT} \quad \Rightarrow \quad I - P_s \left.\frac{dR}{dT}\right|_{T=T^{(0)}} P_s$$

※ Rotation is necessary to avoid zero eigen value

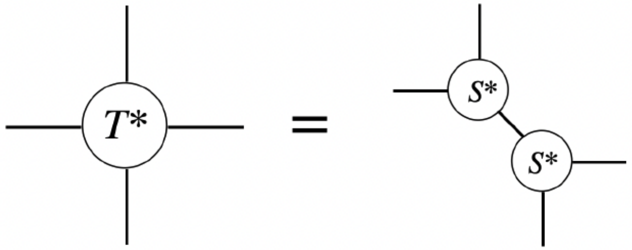


While other methods (i.e., “shooting methods”) tend to go away from the fixed point, this method go toward it by design.)

A. Ueda: arXiv:2401.18068v1 (PhD Thesis)

Elements of the universal tensor in 2D

The elements of fixed-point tensor are the universal correlation function.



$$\frac{S^*_{\alpha\beta\gamma}}{S^*_{111}} = \langle \phi_\alpha(-x_S)\phi_\beta(ix_S)\phi_\gamma(0)\rangle_{\text{pl}}$$

$$x_S = e^{\pi/4}$$

$$x_T = e^{\pi/2}/2$$

$$\frac{T^*_{\alpha\beta\gamma\delta}}{T^*_{1111}} = \langle \phi_\alpha(-x_T)\phi_\beta(ix_T)\phi_\gamma(x_T)\phi_\delta(-ix_T)\rangle_{\text{pl}}$$

※ The counterpart of 3D or higher is not known.

Toward 3D

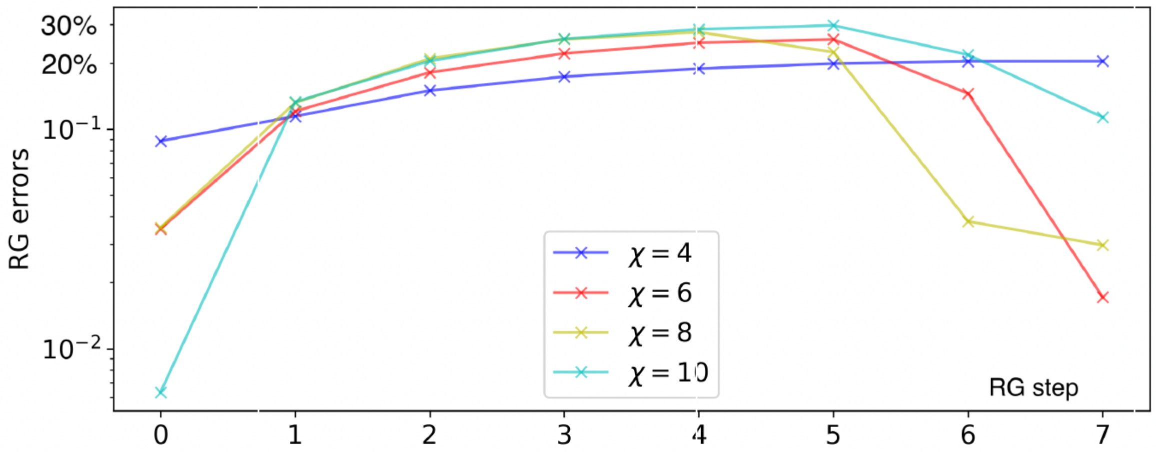
The information mediated by the corner (2D) or the edge (3D) caused by the CDL structure.

$$S \propto L^0 = b^0 \to \text{const} \qquad (2\text{D})$$

$$S \propto L^1 = b^g \to \infty \qquad (3\text{D})$$

While the loop filtering is luxury in 2D, it's necessity in 3D.

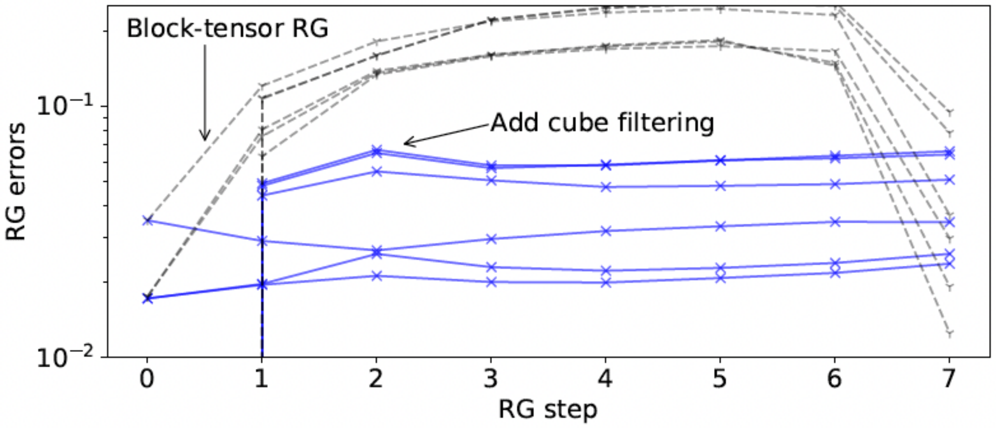
Relative truncation error in RG steps. (T=Tc(χ))



Around n=5 with χ=10, elements of the tensor with almost 30% in amplitude are discarded.

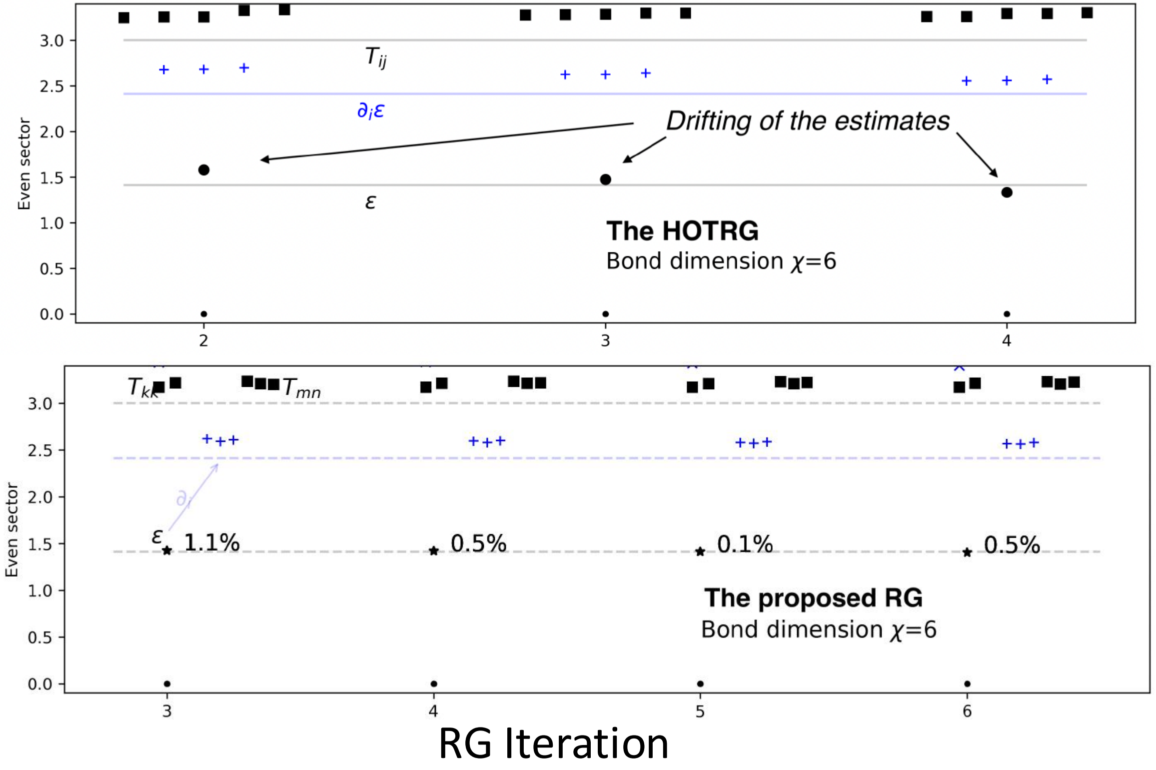
Lyu, NK: 2311.05891v2; PRE 111 (5), 054140 (2025)

Scaling Dimensions in 3D



Entanglement filtering is necessary.

Scaling dimensions in the even parity sector
(obtained by the linearized RG)



Lyu, NK: PRE 111 (5), 054140 (2025)

Summary

- The power of TN methods is the most prominent in its application to real-space RG, and the TN community has made a big progress in real-space RG in the last decade.
- Recently developed methods do not rely on the formula specific to 2D-CFT, paving the way to 3D or higher.
- It is a big mystery why we could have obtained good CFT information without using full conformal symmetry. (We only used dilatation and discrete rotations.)

END