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Two-dimensional two-color lattice QCD with Grassmann tensor renormalization group

Tensor Network 2025 @ 筑波大学 東京キャンパス

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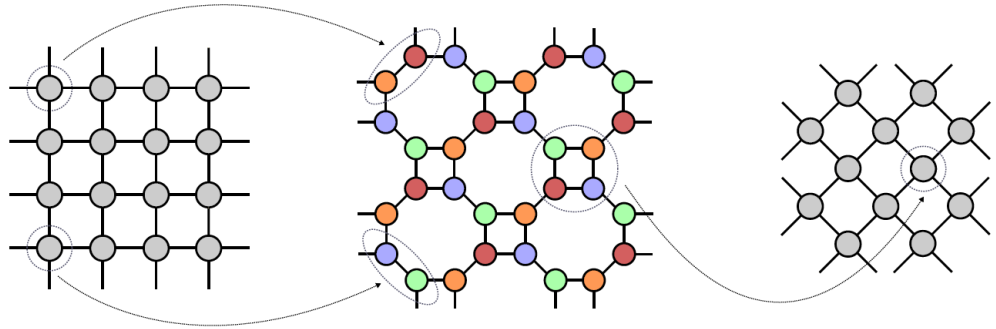
Grassmann TRG approach to lattice field theories

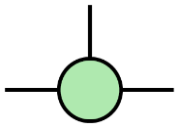
- Tensor network methods are gaining attractions in the research of lattice field theories because they are free from the sign problem, allowing direct simulation at finite density

- Lagrangian approach:
  - Tensor network representation for the partition function
  - TRG algorithm to contract the tensor network



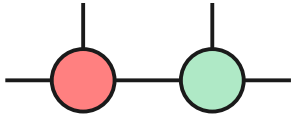
Tensor Renormalization Group (TRG)
[Levin, M., & Nave, C. P. (2007). PRL, 99(12), 120601]

- Grassmann tensor formulation:



Occupation: 0 or 1

$$\mathcal{T}_{i,j,k} = T_{i,j,k}\,\theta_1^{f(i)}\,\theta_2^{f(j)}\,\theta_3^{f(k)}$$



$$\begin{aligned}\mathcal{Y}_{i,j,l,m} &= \mathcal{T}_{i,j,k}\mathcal{G}_{k,l,m} \\ &\equiv \sum_k \int \mathrm{d}\eta_1\,\mathrm{d}\theta_3\,\mathrm{e}^{-\eta_1\theta_3}\,T_{i,j,k}\,G_{k,l,m}\,\theta_1^{f(i)}\,\theta_2^{f(j)}\,\boxed{\theta_3^{f(k)}\,\eta_1^{f'(k)}}\,\eta_2^{f'(l)}\,\eta_3^{f'(m)} \\ &= Y_{i,j,l,m}\,\theta_1^{f(i)}\,\theta_2^{f(j)}\,\eta_2^{f'(l)}\,\eta_3^{f'(m)}\end{aligned}$$

[Akiyama, S., Meurice, Y., & Sakai, R. (2024). Journal of Physics: Condensed Matter, 36(34), 343002.]

- Applications to lattice field theories:

[Shimizu, Y., & Kuramashi, Y., PRD 90.1 (2014): 014508]
[Kanno, H.+, arXiv:2412.08959] [Akiyama, S.+, JHEP 2021(1), 1-17]
[Shimizu, Y., & Kuramashi, Y., PRD 90.7 (2014): 074503.]
[Takeda, S., & Yoshimura, Y., PTEP 2015(4), 043B01]
[Yosprakob, A+, JHEP 2023(11), 1-32]
[Bloch, J. & Lohmayer, R., Nucl. Phys. B 986 (2023) 116032]
[Asaduzzaman, M. +, JHEP 05 (2024) 195] …

(1+1)-D lattice two-color QCD with staggered fermions

What we calculate with TRG

$$Z = \int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\, \mathrm{e}^{-S}$$

Parameters: m, β, μ, λ

$$S = S_f + S_g + S_\lambda$$

Wilson's gauge action

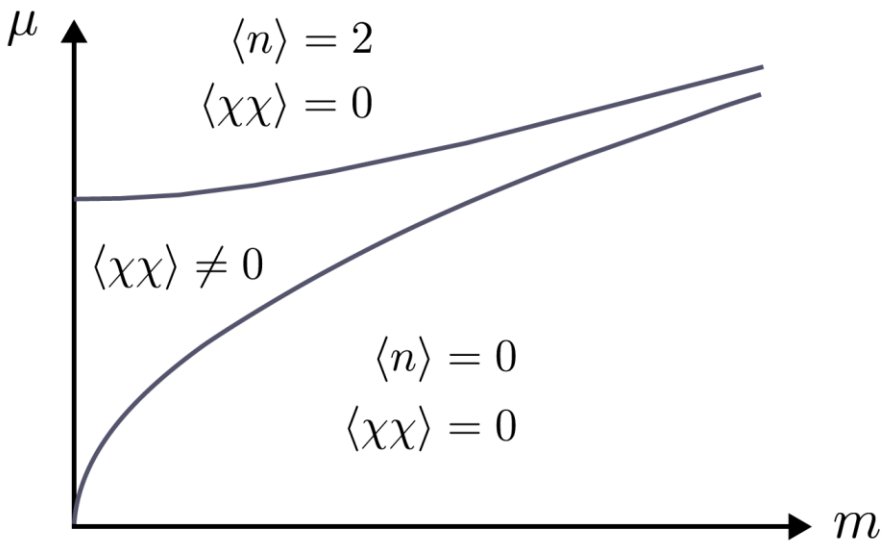
Fermion hopping term + mass term

Diquark source term

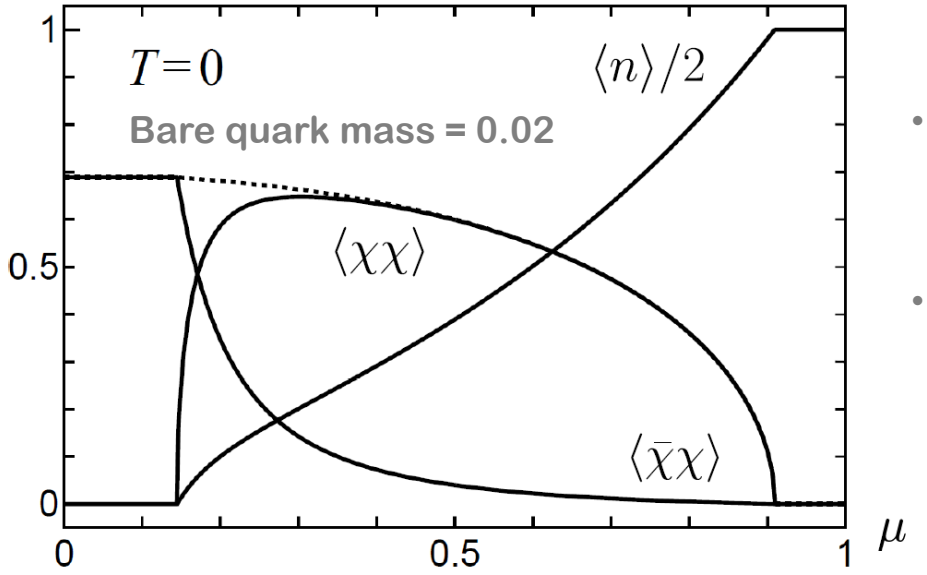
$$S_\lambda = \frac{\lambda}{2}\sum_n \left[\chi^T(n)\sigma_2\chi(n) + \bar{\chi}(n)\sigma_2\bar{\chi}^T(n)\right]$$

$$\langle\chi\chi\rangle \equiv \frac{1}{2V}\int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\sum_n \left(\chi^T\sigma_2\chi + \bar{\chi}\sigma_2\bar{\chi}^T\right)\mathrm{e}^{-S}$$

Phase structure of the (3+1)-D theory



[Y. Nishida+, Phys. Rept. 398 (2004) 281–300]



(3+1)-D infinite coupling two-color QCD with staggered fermions

- Finite diquark condensate due to the SSB of U(1) symmetry
- Lattice artifact comes into play at large chemical potential

Tensor network representation

$$Z = \int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\, \mathrm{e}^{-S}$$

- **The fermion determinant can be expressed as the trace of a Grassmann tensor network by re-writing every hopping term in the lattice action into a Grassmann integral of two auxiliary Grassmann fields**

  [Akiyama, S., & Kadoh, D., JHEP, 2021(10), 1-16]

  - Fermionic bond dimension: $2^{2(\text{number of color})}$

    [M. Asaduzzaman+, JHEP 05 (2024) 195]: $2^{2(\text{number of color})^2}$

  - The Grassmann tensor network in this step depends on the configuration of the gauge fields. However, the gauge group integration can be done exactly in the infinite coupling limit.

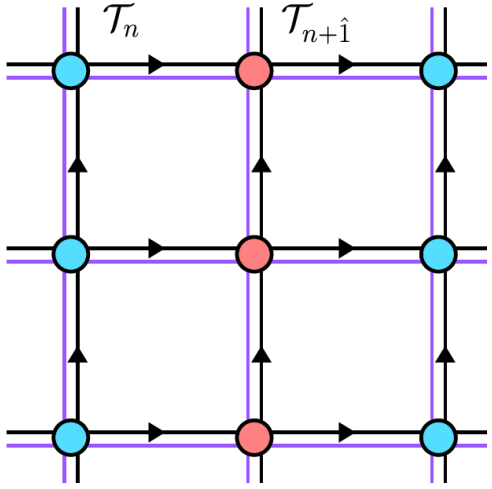
- **At finite couplings, the gauge group integration is approximated by a summation of terms where the integrand is evaluated using group elements sampled uniformly from the group manifold** [Fukuma, M.+, PTEP, 2021(12),123B03]

$$\int \mathrm{d}U\, f(U) \simeq \frac{1}{K}\sum_{i=1}^{K} f(U_i)$$

Sample size

  - The trace of the resulting Grassmann tensor network approximates the partition function

  - The bond dimension of the initial tensors is 16K



Initial tensor compression

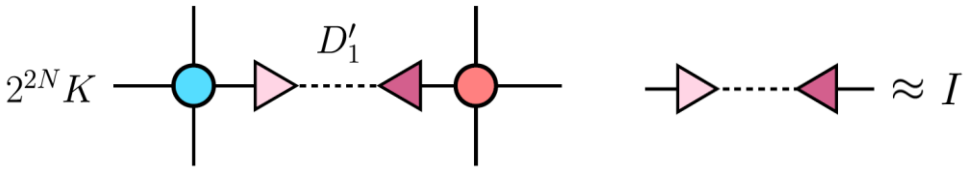
[Adachi, D.+, PRB, 105(6), L060402 (2022)]
[Akiyama, S., JHEP, 2022(11), 1-14]

N = no. of color
K = sample size

- We use bond-weighted tensor renormalization group to coarse-grain the tensor network and reach the thermodynamic limit

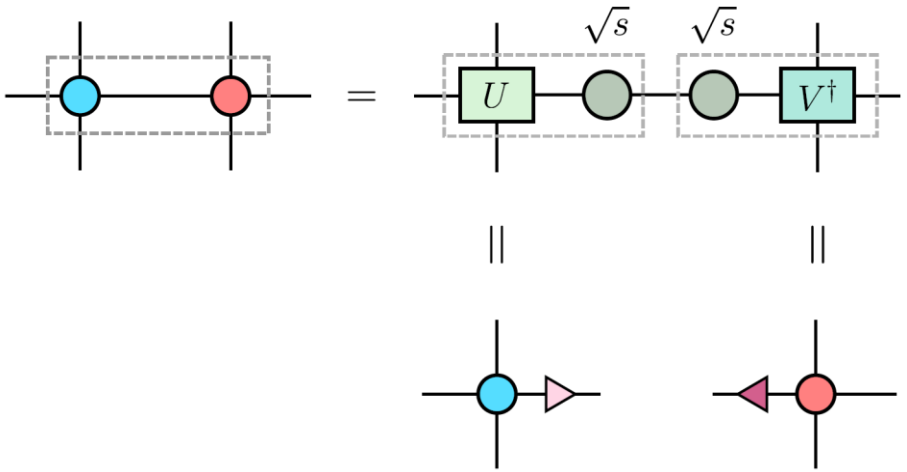
- The choice of bond dimension cutoff D in TRG algorithms depends on the bond dimension of initial tensors. For our case (two-color QCD), the initial bond dimension can have an order of 10^2 (=16K)

- Compression of initial tensors is needed before TRG:



insert a pair of squeezers (a good approximation of identity) on every bond of the tensor network

- The insertion of squeezers is equivalent to a truncated SVD on the following rank 6 tensor



How to determine the bond dimension after compression (how many singular values are kept)?

Bond dimension after compression

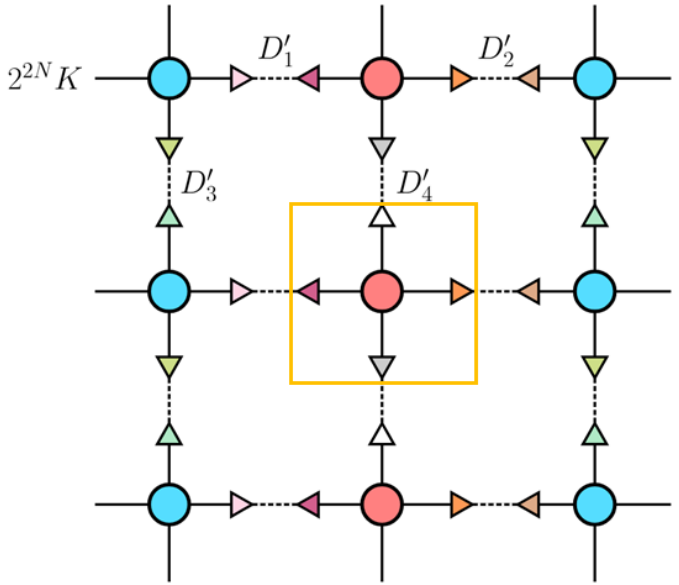
$$\frac{\sum_{y=1}^{D'} s_y^2}{\sum_{y=1}^{2^{2N}K} s_y^2} \geq r$$

Ratio parameter

Initial bond dimension

Efficiency of compression

N = no. of color
K = sample size



$$\frac{\sum_{y=1}^{D'} s_y^2}{\sum_{y=1}^{2^{2N}K} s_y^2} \geq r$$

Bond dimension after compression

Ratio parameter

Initial bond dimension

$m = 0.1,\ \beta = 1.6,\ \mu = 0.4,\ \lambda = 0,\ K = 14$

r	D'_1	D'_2	D'_3	D'_4	compression rate
1	224	224	224	224	100%
0.99999	148	148	143	143	17.8%
0.99995	122	122	118	118	8.23%
0.9999	110	110	105	105	5.30%
0.9995	80	80	79	79	1.59%
0.999	70	70	67	67	0.874%
0.99	35	35	33	33	0.0530%

Accuracy of compression

$m = 0.1,\ \beta = 0.8,\ \mu = 0.4,\ \lambda = 0,\ K = 14$

r	D'_1	D'_2	D'_3	D'_4	compression rate
1	224	224	224	224	100%
0.99999	86	86	84	84	2.07%
0.99995	68	68	66	66	0.800%
0.9999	61	61	59	59	0.514%
0.9995	46	46	43	43	0.155%
0.999	39	39	37	37	0.0827%
0.99	19	19	19	19	0.00518%

Calculation of observables

Free energy density:

$$f = \ln Z/V$$ What we calculate directly with TRG

Quark number density:

$$\langle n \rangle = \frac{\partial f}{\partial \mu} \simeq \frac{f(\mu + \Delta\mu) - f(\mu)}{\Delta\mu}$$

$\Delta\mu = 0.04$ for $m = 0.1$

$\Delta\mu = 0.02$ for $m = 1$

Chiral condensate:

$$\langle \bar{\chi}\chi \rangle = \frac{\partial f}{\partial m} \simeq \frac{f(m + \Delta m) - f(m)}{\Delta m}$$

$\Delta m = 10^{-4}$

Diquark condensate:

$$\langle \chi\chi \rangle = \frac{\partial f}{\partial \lambda} \simeq \frac{f(\lambda + \Delta\lambda) - f(\lambda)}{\Delta\lambda}$$

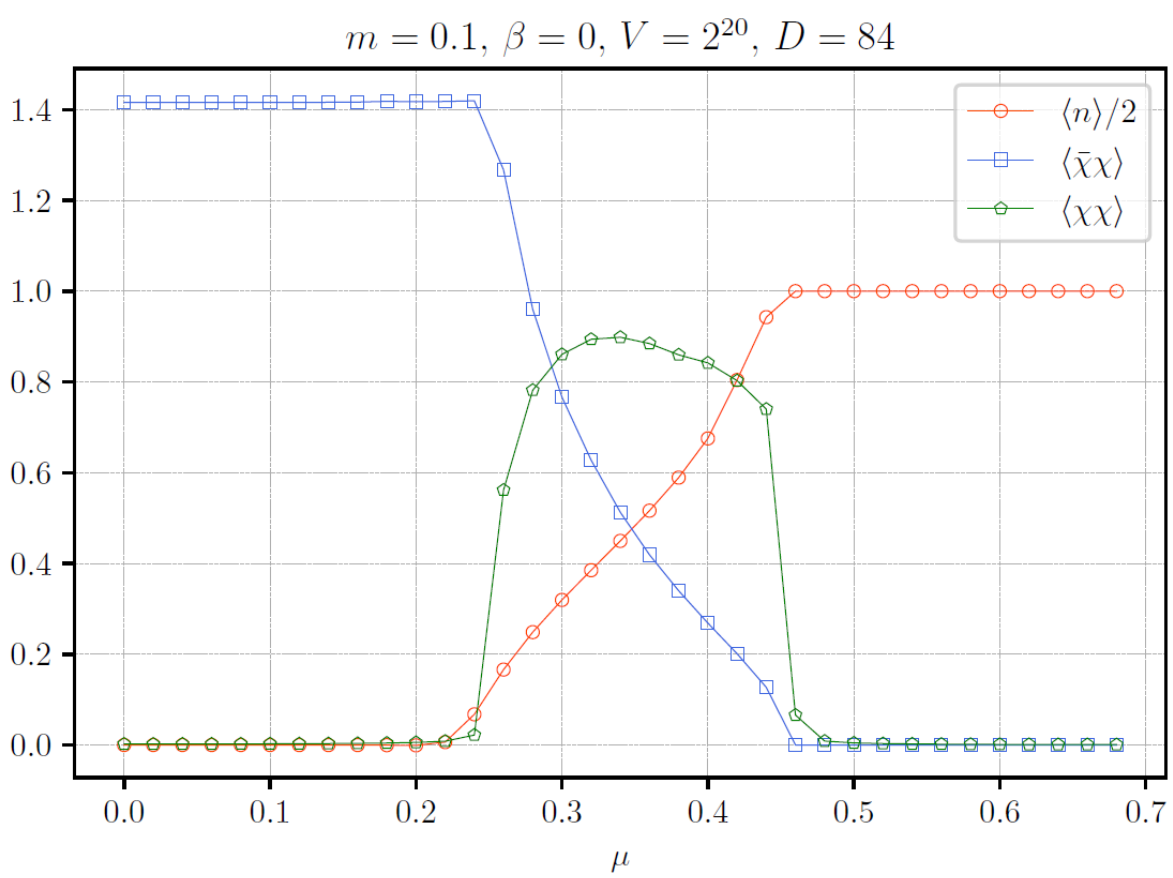
$\lambda = \Delta\lambda = 10^{-4}$

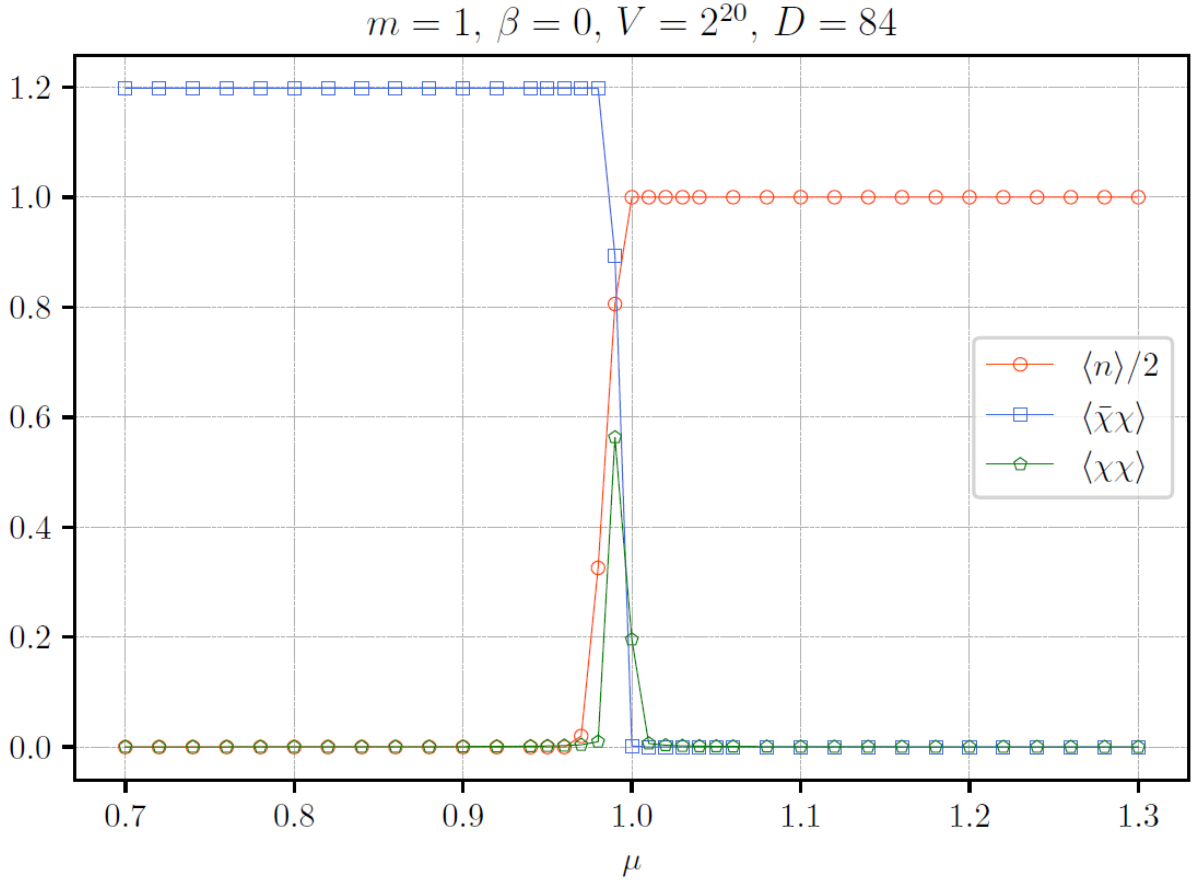
Remarks:

- **We always consider finite m and/or λ, which breaks the U(1) symmetries explicitly**
- **Otherwise, chiral condensate and diquark condensate cannot have a finite value in 2D**

Numerical results: infinite coupling limit

Initial bond dimension is 16
no initial tensor compression here





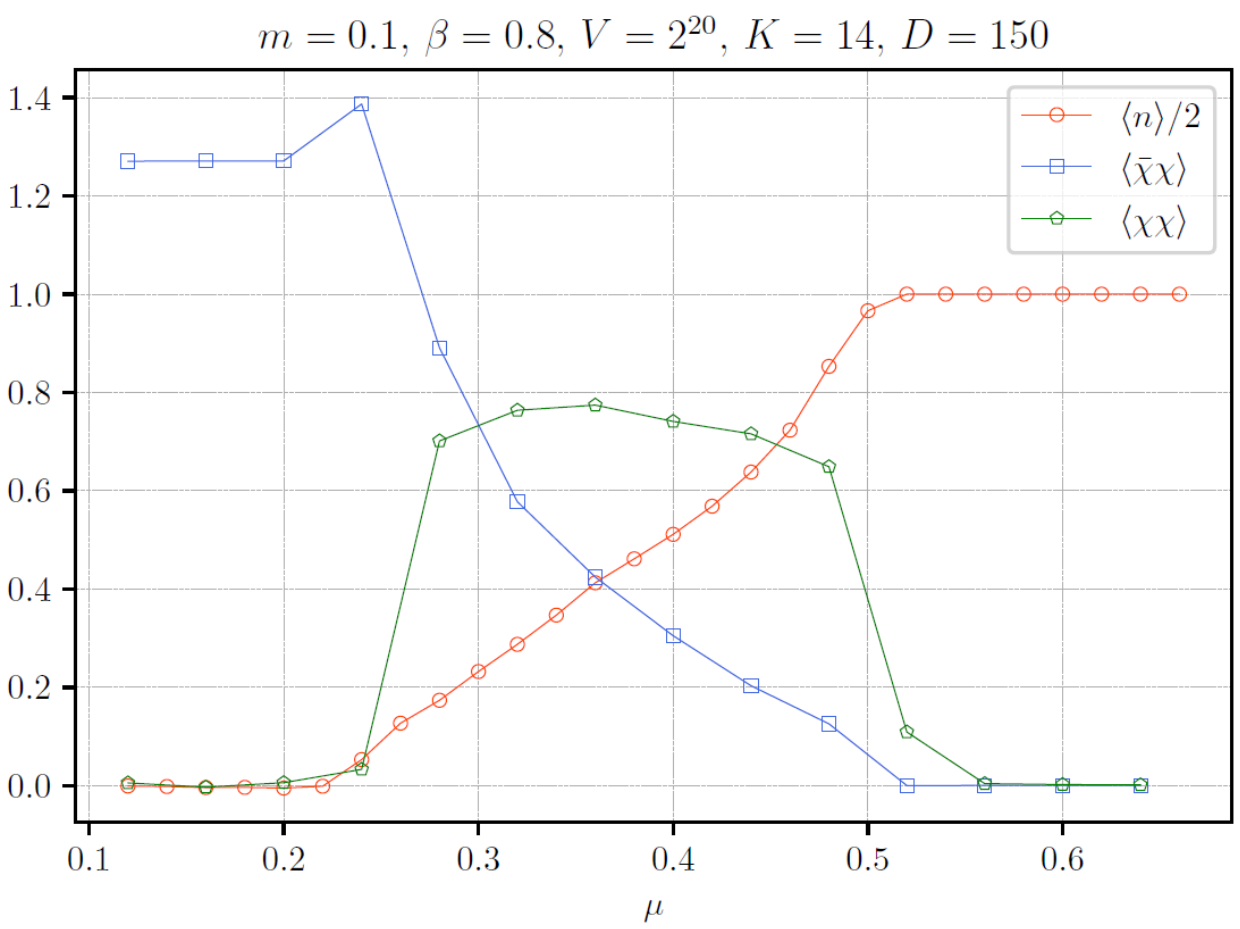
@ m = 0.1: an intermediate phase is observed in a finite region of μ

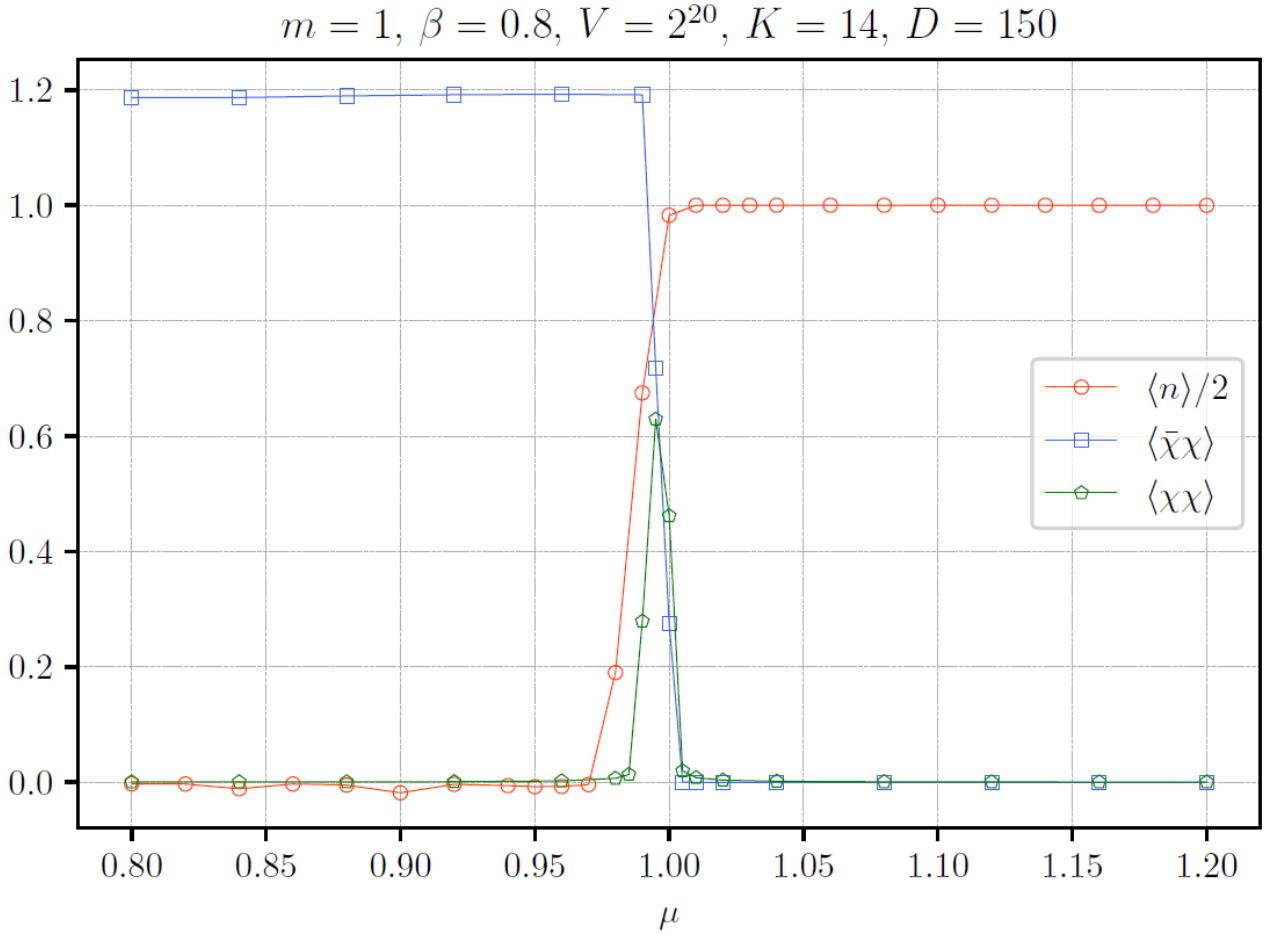
@ m = 1: a sharp transition is seen, and the intermediate phase becomes a very narrow region in μ

- The qualitative behavior of the observables in 2D (with explicitly broken symmetries) is like the one reported in a mean-field study of the (3+1)-D theory, where spontaneous symmetry breaking occurs
  [Y. Nishida+, Phys. Rept. 398 (2004) 281–300]

Numerical results: $\beta = 0.8$

K = 14

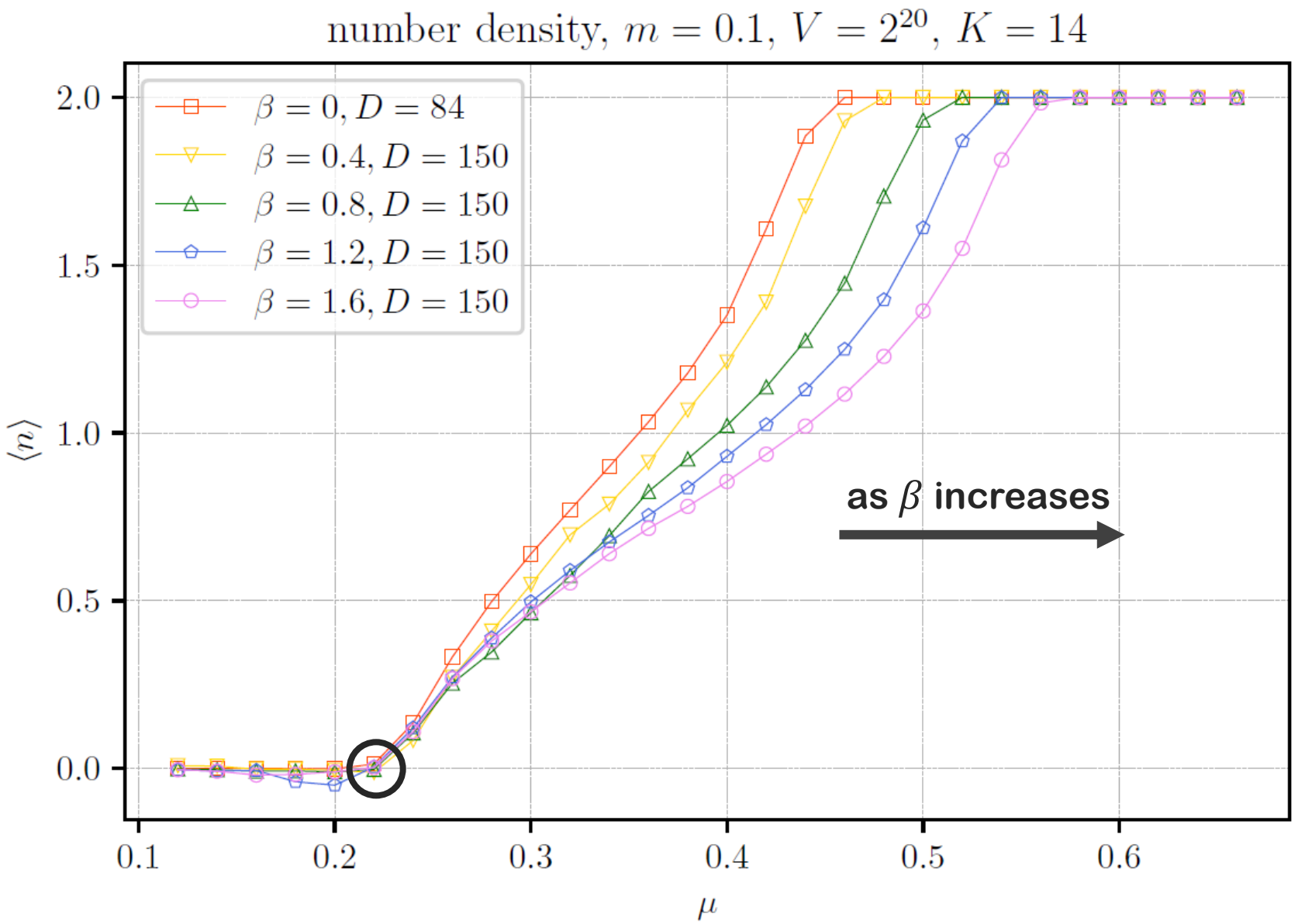




- The behavior at finite coupling is like that at infinite coupling
- As β becomes nonzero, the intermediate phase becomes broader at m = 0.1

$\beta = 0$ $0.22 \leq \mu \leq 0.46$ $\beta = 0.8$ $0.22 \leq \mu \leq 0.52$

β dependence of transition points



- The first transition point (the one at a smaller μ) seems to be robust against β
- The second transition point locates at larger chemical potential as β increases
- $\langle n \rangle$ does not saturate in regions of larger chemical potential as the gauge interaction is weakened, approaching the continuum limit

Summary

- This is a TRG study on non-Abelian gauge theory coupled with standard staggered fermions at finite density and finite coupling
- Tensor network calculation for this kind of theories is computationally challenging because of the very large initial bond dimension
- We introduce an efficient initial tensor compression scheme to deal with this issue
- TRG enables the calculation of important physical quantities at the infinite coupling limit and finite β regime
- Future directions:
  1) TRG calculations of multi-flavor theories
  2) Higher-dimensional calculations
  3) More detailed study on the intermediate phase

Thank you for listening!