Complex entanglement entropy for complex conformal field theory

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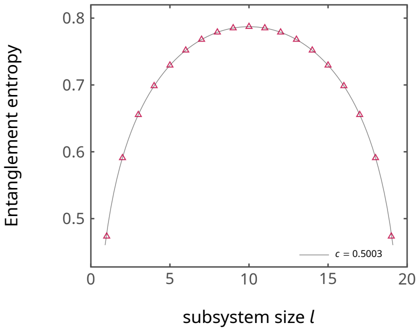
Entanglement entropy and CFT

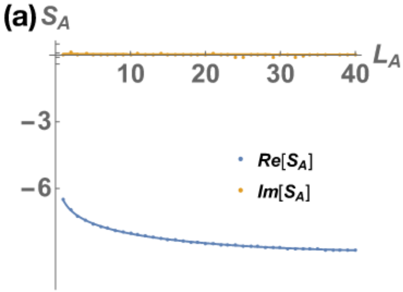
$$S \sim \frac{c}{3} \log \left(\frac{L}{\pi} \sin \frac{\pi l}{L} \right) \quad \begin{cases} c : \text{central charge} \\ L : \text{total system size} \\ l : \text{subsystem size} \end{cases}$$

[Holzhey-Larsen-Wilczek, 1994]
[Calabrese-Cardy 2004]

ex.) Ising model $(c = 0.5)$

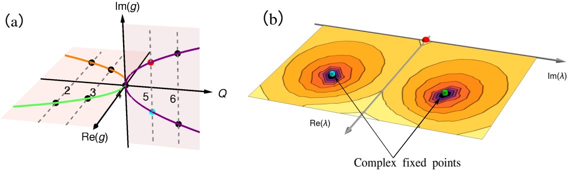


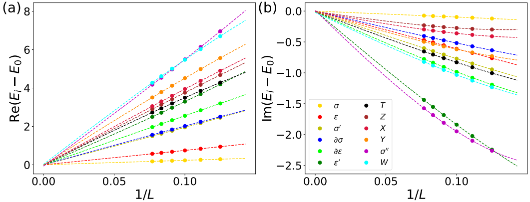
ex.) non-Hermitian SSH model $(c = -2)$
[Chang-You-Wen-Ryu, 2020]



- $Q \leq 4$: second-order
- $Q > 4$: weakly first-order

But, expansion of the parameter space on a **complex plane** brings **complex CFT**.
[Gorbenko-Rychkov-Zan, 2018]
[Tang-Ma-Tang-He-Zhu, 2024]





OPE	non-Hermitian 5-Potts	Analytical Continuation
$C_{\epsilon\epsilon\epsilon'}$	$0.8781(72) - 0.1433(21)i$	$0.8791 - 0.1404i$
$C_{\epsilon'\epsilon'\epsilon'}$	$2.2591(77) - 1.1916(39)i$	$2.2687 - 1.1967i$
$C_{\epsilon\epsilon'\epsilon''}$	$0.804(3) - 0.220(11)i$	$0.8318 - 0.2027i$
$C_{\epsilon''\epsilon''\epsilon'}$	$3.886(96) - 3.369(33)i$	$3.9261 - 3.3261i$
$C_{\sigma\sigma\epsilon'}$	$0.0658(15) + 0.0513(10)i$	NA
$C_{\sigma\sigma\epsilon}$	$0.7170(6) + 0.1558(1)i$	NA
$C_{\sigma\sigma'\epsilon}$	$0.7520(15) - 0.0831(6)i$	NA
$C_{\sigma\sigma'\epsilon'}$	$0.6062(4) + 0.1664(10)i$	NA
$C_{\sigma\sigma''\epsilon'}$	$0.6709(45) - 0.1250(21)i$	NA

Energy spectrum $\leftrightarrow$ Conformal towers

Correlation function $\leftrightarrow$ OPE coefficients

Lattice value matches well with complex CFT prediction! How about the **entanglement entropy**?

Potts model with perturbation

$$H_{\text{NH-Potts}}(J,h,\lambda) = H_0(J,h) + H_1(\lambda),$$

$$H_0(J,h) = -\sum_{i=1}^{L}\sum_{k=1}^{Q-1}\left[J(\sigma_i^\dagger\sigma_{i+1})^k + h\tau_i^k\right],$$

$$H_1(\lambda) = \lambda\sum_{i=1}^{L}\sum_{k_1,k_2=1}^{Q-1}\left[(\tau_i^{k_1}+\tau_{i+1}^{k_1})(\sigma_i^\dagger\sigma_{i+1})^{k_2} + \text{h.c.}\right],$$

With $Q=5$, at

- $J_c = 1$,
- $h_c = 1$,
- $\lambda_c = 0.079 \pm 0.060\mathrm{i}$,

the model exhibits complex CFT with

$$c \approx 1.13755 \mp 0.0210687\mathrm{i}.$$

where

$$\sigma = \begin{pmatrix} e^{2\pi\mathrm{i}/Q} & & & & \\ & e^{4\pi\mathrm{i}/Q} & & & \\ & & \ddots & & \\ & & & e^{2(Q-1)\pi\mathrm{i}/Q} & \\ & & & & 1 \end{pmatrix}, \tau = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}.$$

Method — DMRG

In order to see the complex behavior, we cannot use the usual density matrix as Hermitian case.

1. calculate left and right eigenvectors using DMRG

$$H\,|\psi\rangle = E\,|\psi\rangle\,, \quad H^\dagger\,|\psi\rangle\rangle = E^*\,|\psi\rangle\rangle$$

2. construct generalized reduced density matrix

$$\rho^{\mathrm{RL}} := \frac{|\psi\rangle\,\langle\langle\psi|}{\langle\langle\psi|\psi\rangle}, \quad \rho_A^{\mathrm{RL}} := \mathrm{tr}_B\,\rho^{\mathrm{RL}}$$

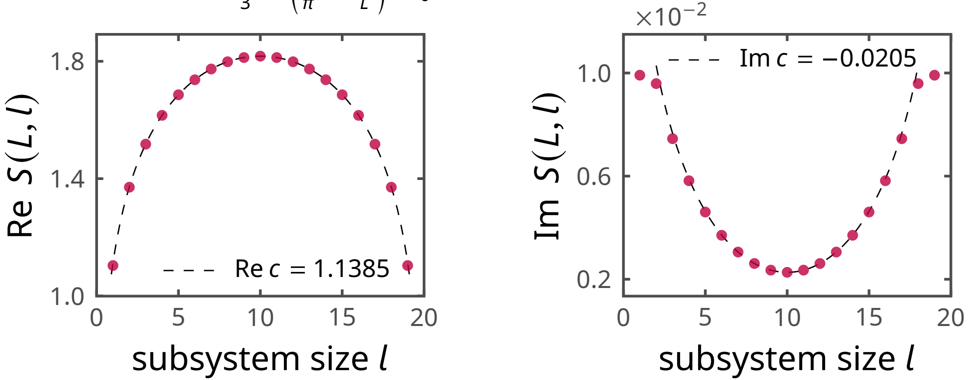
3. calculate the von Neumann entropy (**generalized entanglement entropy**)

$$\begin{aligned} S_A &= -\,\mathrm{tr}_A(\rho_A^{\mathrm{RL}}\log\rho_A^{\mathrm{RL}}) \\ &= -\sum_i \lambda_i \log \lambda_i \end{aligned}$$

Numerical Results — Periodic boundary condition

$L = 20,\ D = 400$, **Periodic** boundary condition

Fitting Formula : $S = \frac{c}{3} \log\left(\frac{L}{\pi} \sin \frac{\pi l}{L}\right) + S_0$

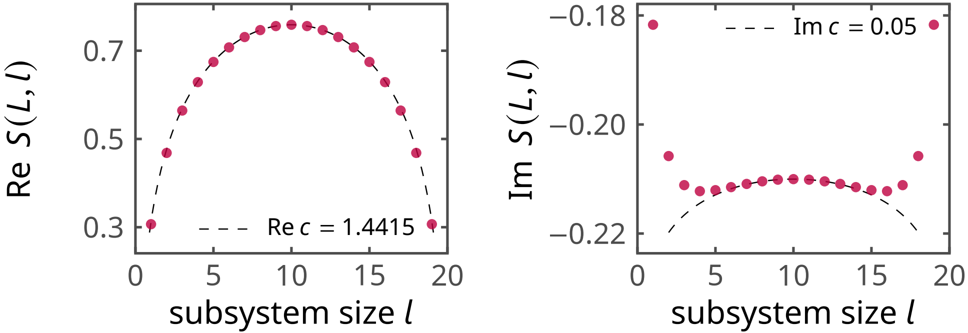


Similar to the analytical value $c \approx 1.13755 - 0.0210687\mathrm{i}$.

Numerical Results — Open boundary condition

$L = 20,\ D = 400$, **Open** boundary condition

Fitting Formula : $S = \frac{c}{6} \log\left(\frac{2L}{\pi} \sin \frac{\pi l}{L}\right) + S_0$



Far from the analytical value $c \approx 1.13755 - 0.0210687i$.

Comparison with other definitions

SVD entropy:

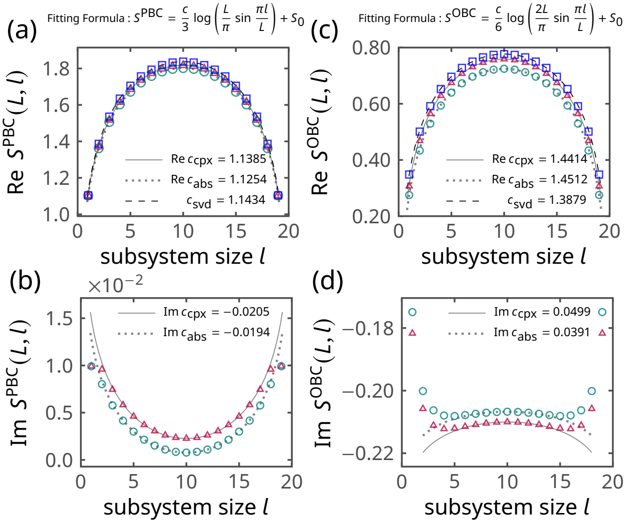
$$\hat{\rho}_{\mathrm{A}}^{\mathrm{SVD}} := \frac{\sqrt{(\hat{\rho}_{\mathrm{A}}^{\mathrm{RL}})^{\dagger}\hat{\rho}_{\mathrm{A}}^{\mathrm{RL}}}}{\mathrm{tr}\sqrt{(\hat{\rho}_{\mathrm{A}}^{\mathrm{RL}})^{\dagger}\hat{\rho}_{\mathrm{A}}^{\mathrm{RL}}}}$$

$$S_A = -\,\mathrm{tr}_A(\rho_A^{\mathrm{SVD}}\log\rho_A^{\mathrm{SVD}})$$

Entropy using absolute value of the eigenvalues:

$$S_{\mathrm{abs}} := -\sum_i \lambda_i \log|\lambda_i|$$

The **generalized entanglement entropy** mostly agrees with the CFT prediction.



Summary and Outlook

Summary

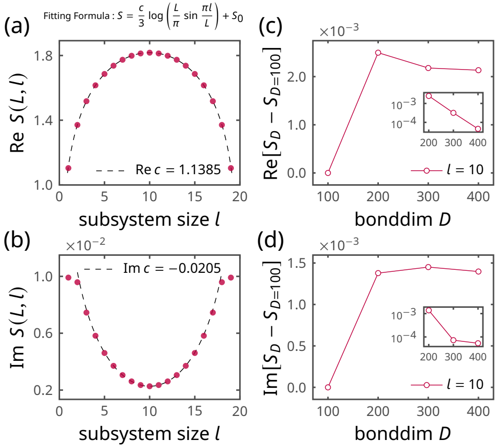
- We can see that non-Hermitian 5-state Potts model obeys complex CFT from the entropy obtained from generalized reduced density matrix.
- Periodic case is consistent with the prediction of complex CFT, but in open case the model is exposed to some kind of boundary effect.
- The generalized entanglement entropy seems to be the appropriate way to measure the entanglement entropy in complex CFT.

Outlook

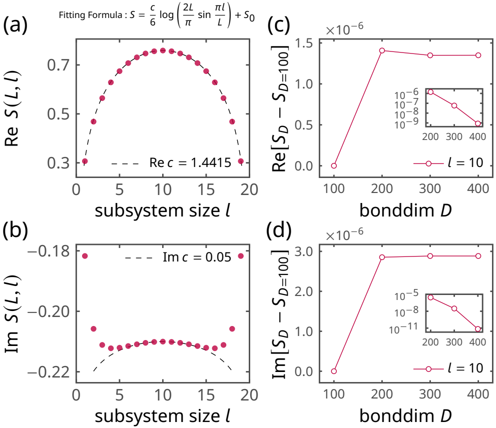
- Entropy scaling with other boundaries (cf. [Tang-Liu-Tang-Zhu, 2024])
- Does something like c-theorem exist in complex CFT?

Appendix — Change of EE with bond dimension

Periodic boundary condition



Open boundary condition



Appendix — Central charge scaling

	$c\,(D=300)$	$c\,(D=400)$
CFT		$1.1376-0.0211\mathrm{i}$
$L=4$	$1.1927+0.0104\mathrm{i}$	$1.1927+0.0104\mathrm{i}$
$L=8$	$1.1429-0.0171\mathrm{i}$	$1.1429-0.0171\mathrm{i}$
$L=12$	$1.1398-0.0196\mathrm{i}$	$1.1397-0.0195\mathrm{i}$
$L=16$	$1.1389-0.0201\mathrm{i}$	$1.1388-0.0203\mathrm{i}$
$L=20$	$1.1388-0.0209\mathrm{i}$	$1.1385-0.0205\mathrm{i}$
$L=24$	$1.1391-0.0208\mathrm{i}$	$1.1384-0.0206\mathrm{i}$
$L=24\ (\hat{\rho}_{\mathrm{A}}^{\mathrm{R}})$	1.1635	1.1633
$L=24\ (\hat{\rho}_{\mathrm{A}}^{\mathrm{SVD}})$	1.1283	1.1431