TOHOKU UNIVERSITY

GPPU GRADUATE PROGRAM ON PHYSICS FOR THE UNIVERSE

宇宙創成物理学国際共同大学院

大規模計算に向けた4次元テンソル繰り込み群の改良とその応用

IMPROVEMENTS OF FOUR-DIMENSIONAL TENSOR RENORMALIZATION GROUP FOR LARGE-SCALE COMPUTATION AND ITS APPLICATIONS

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[Y. Sugimoto and S. Sasaki, PoS LATTICE2024 (2025) 038]
[Y. Sugimoto. 素粒子論研究・電子版 Vol. 44 (2025) No. 3]

ABSTRACT

- 4D **finite-density QCD** is important for physics of neutron star
  - ✕ Difficult with Monte-Carlo
- **Tensor Renormalization Group** is a candidate for alternative algorithm
- However, the computational cost is extremely high in higher dimensions

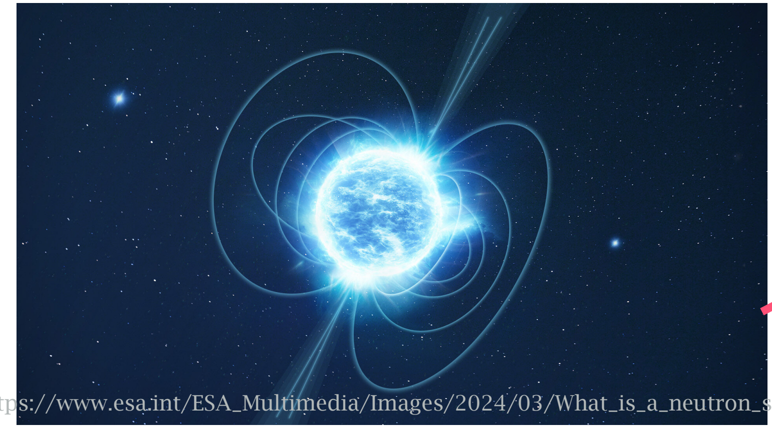
↓

I have proposed faster algorithm for 4D TRG method

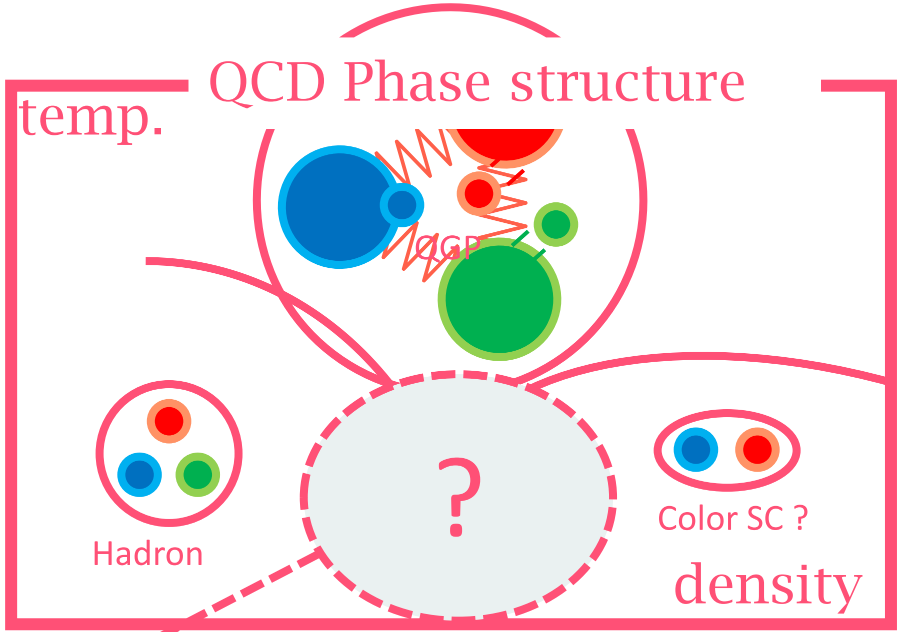
BACKGROUND

GOAL : First-principles calculations of finite density QCD

- **QCD** : A 4D Theory of the Strong Force
  - Interactions between quarks and gluons
    → color confinement , hadron mass …
  - Related to neutron star at **finite density**

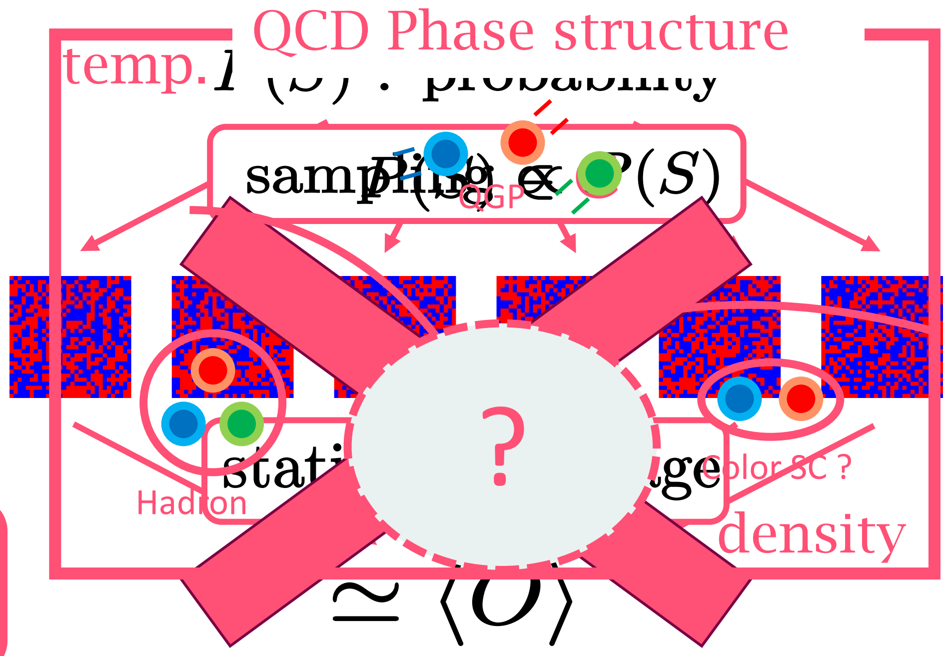


https://www.esa.int/ESA_Multimedia/Images/2024/03/What_is_a_neutron_star



SIGN PROBLEM

- Lattice QCD can't be used at finite density
  - MC methods rely on stochastic sampling
  - Complex weight at finite density
  - QCD phase diagram is not calculable



sign problem

alternative algorithms ?

→ **Tensor Renormalization Group (TRG)**

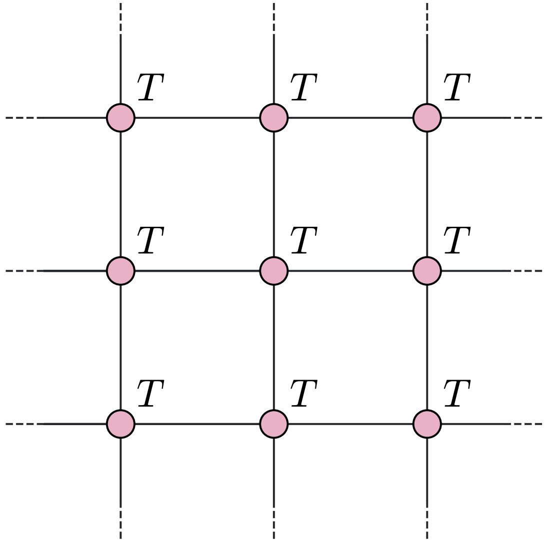
[M. Levin and C. P. Nave, (2007).]

ABOUT TRG

[M. Levin and C. P. Nave, (2007).] [Xie et al, (2012).]

- TRG : Approximate partition function Z by SVD
  - Proposed in 2D, later extended to higher dimensions (HOTRG)
- Z can be expressed by tensor network (contraction of tensors)

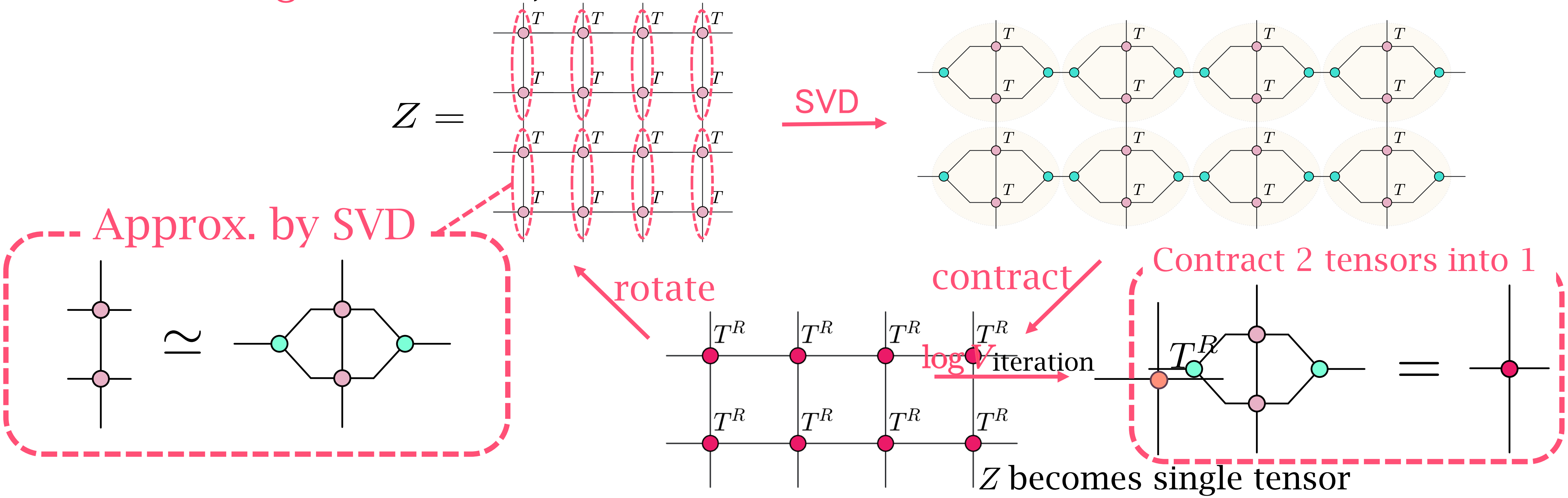
$$Z = \sum_{i,j,k,l,m,\ldots}^{D} T_{ijkl}T_{mnoj}T_{krst}T_{opqr}\cdots =$$



HOTRG [Xie et al, (2012).]

- Renormalize 2 tensors into one
- “Coarse-Graining” of lattice
- After $\log V$ iteration, # of tensors becomes 1

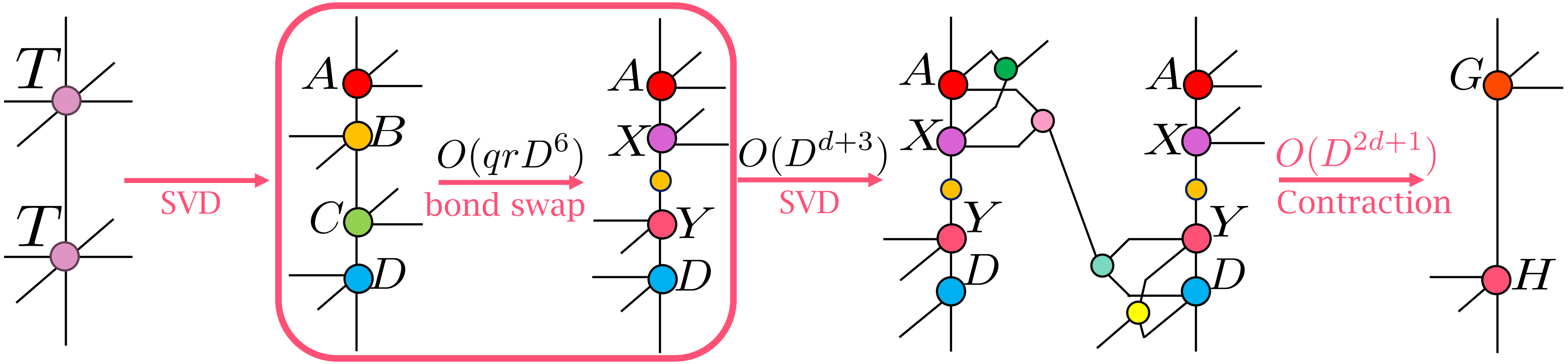
Huge cost ! $O(D^{4d-1})$



ATRG

1. D. Adachi, T. Okubo, and S. Todo, (2020).
2. S. Akiyama and Y. Kuramashi, (2023).

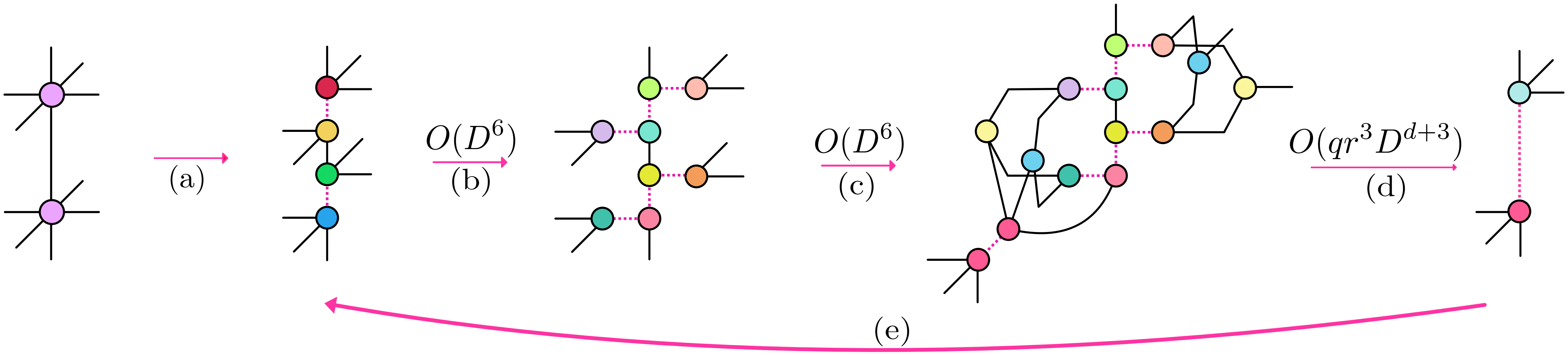
- ATRG[1] is lighter algorithm of the HOTRG
- Cost reduction $O(D^{4d-1}) \to O(D^{2d+1})$
- Some applications for 4D[2]
- Still large cost → difficult to increase D



TRIAD-MDTRG

[K. Nakayama, (2023).]

- Triad-MDTRG : decompose TT (not local) into 3-legs tensors (speedup)
- oversample (use $D \to rD$ singular values for internal line)
- Contraction via RSVD (speedup)
- Accuracy : almost equivalent to HOTRG



ALGORITHMS IN 4D

1. D. Adachi, T. Okubo, and S. Todo, (2020).
2. K. Nakayama, (2023).

HOTRG

- cost : $O(D^{15})$ 😱
- Original

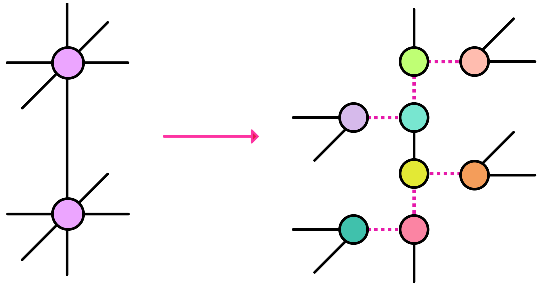
ATRG[1]

- cost $O(D^9)$
  - Bond Swap
- Application in 4D
- ✕ Large D is difficult

Triad-MDTRG[2]

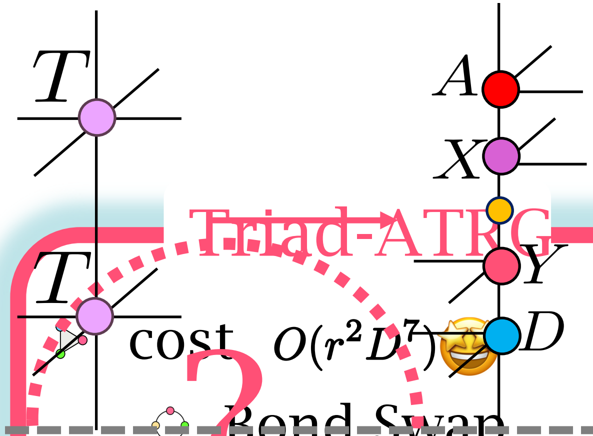
- cost $O(r^3D^8)$
  - Triad rep.
  - oversample
- Good accuracy
- ✕ No application in 4D

All tensors are 3-leg



Use $D \to rD$ singular values

RSVD sub network



Triad-ATRG

- cost $O(r^2D^7)$ 🤩
  - Bond Swap
  - Triad rep.
  - Oversample

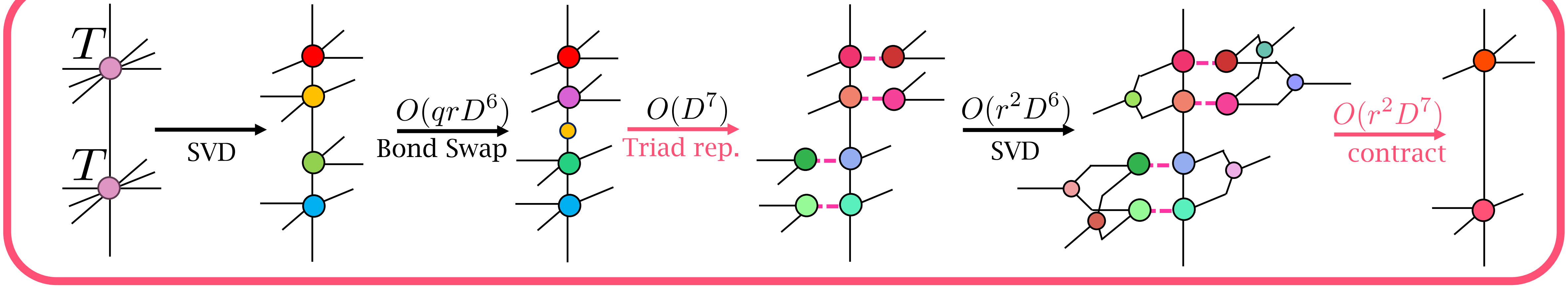
This Work

TRIAD-ATRG

1. D. Adachi, T. Okubo, and S. Todo, (2020).
2. K. Nakayama, (2023).

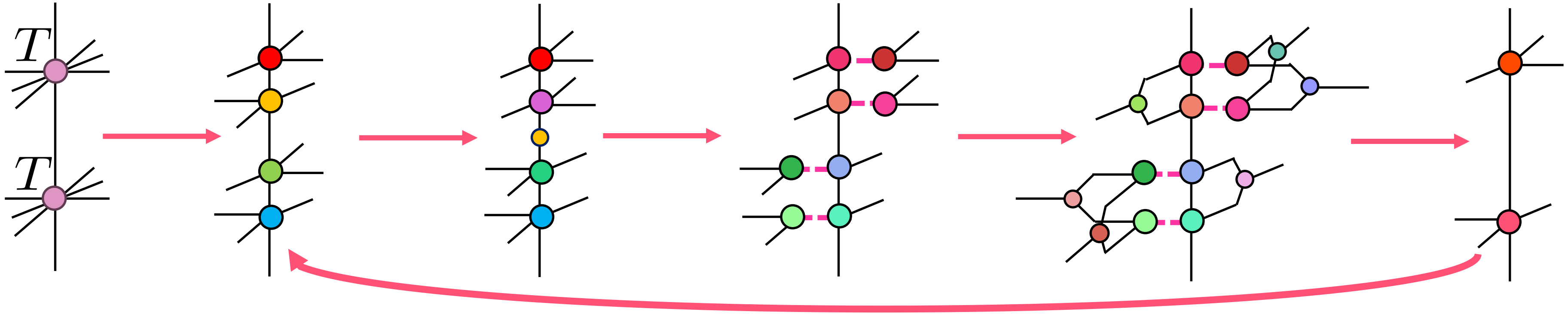
- Applying Triad rep. to 4D ATRG[1], reduced the cost $O(D^9) \to O(r^2 D^7)$
- Maintaining the accuracy from ATRG by using MDTRG[2] technique
- oversampling parameter[2] r : The larger the r, the closer to ATRG (r=D)

Triad-ATRG



NUMERICAL RESULTS IN 4D ISING MODEL

- propose Triad-ATRG method



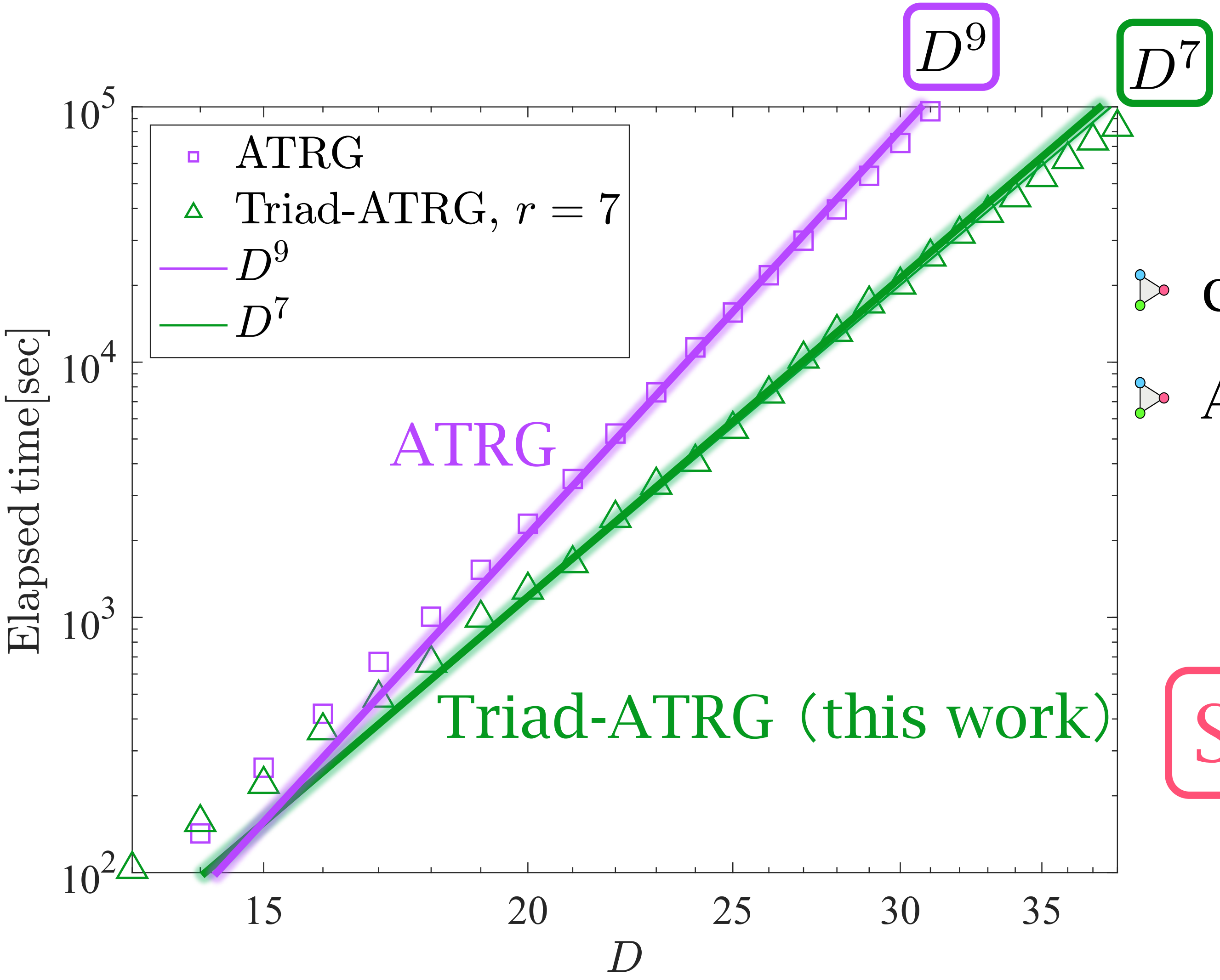
- Benchmarking with 4D Ising model

$$H = -\sum_{n}\prod_{\mu}^{4}\sigma_n\sigma_{n+\hat{\mu}} - h\sum_{n}\sigma_n, \quad Z = \sum_{\{\sigma\}} e^{-\beta H}$$

- Critical point
- Previous research : Monte Carlo, HOTRG, ATRG

[S. Akiyama, Y. Kuramashi, T. Yamashita, and Y. Yoshimura, (2020).]
[P. H. Lundow and K. Markström, (2023).]

SCALING OF COMPUTATIONAL TIME

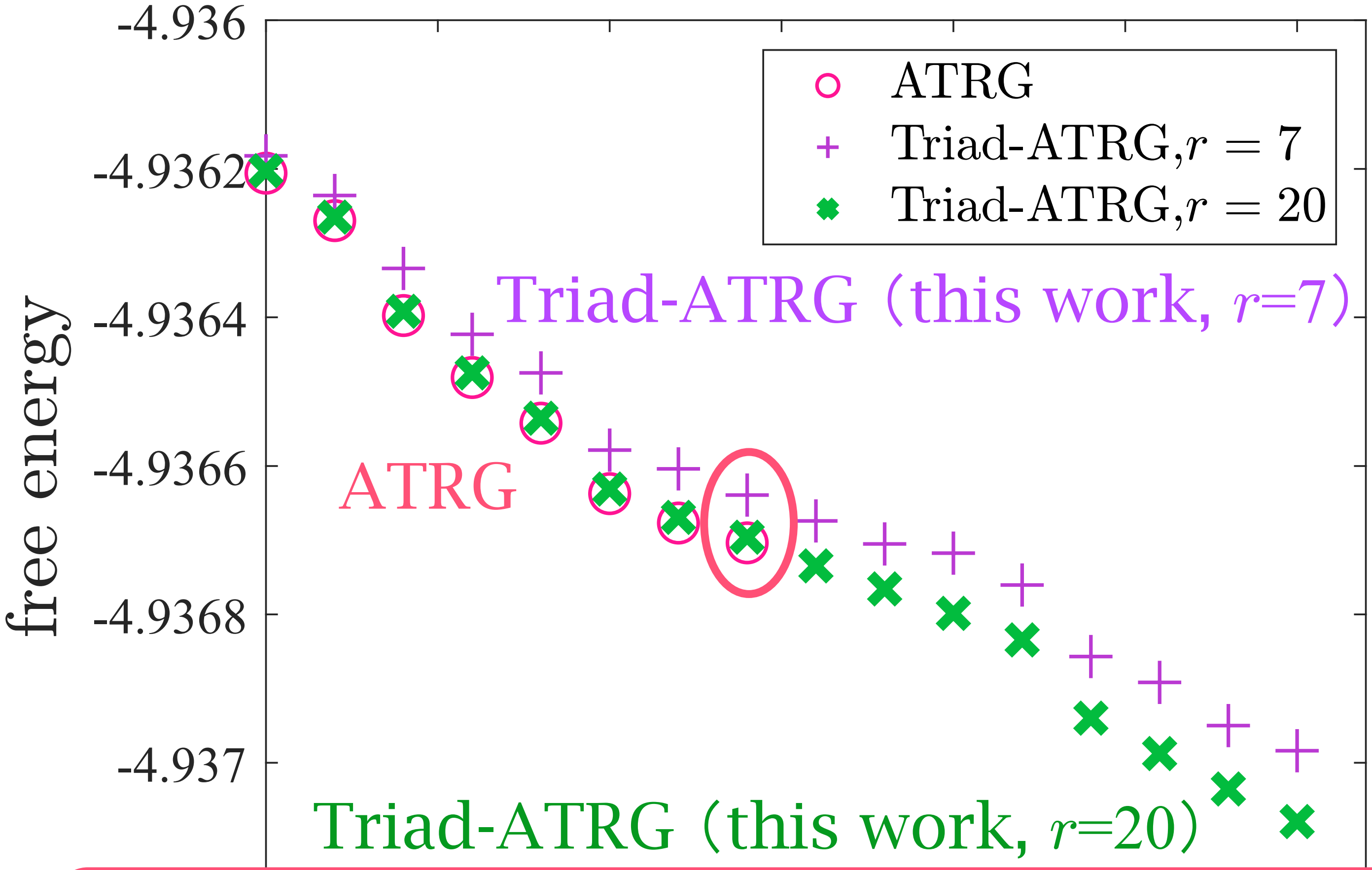


- comparison of computational time
- ATRG : $O(D^9)$,Triad-ATRG : $O(D^7)$

Significant cost reduction !

FREE ENERGY

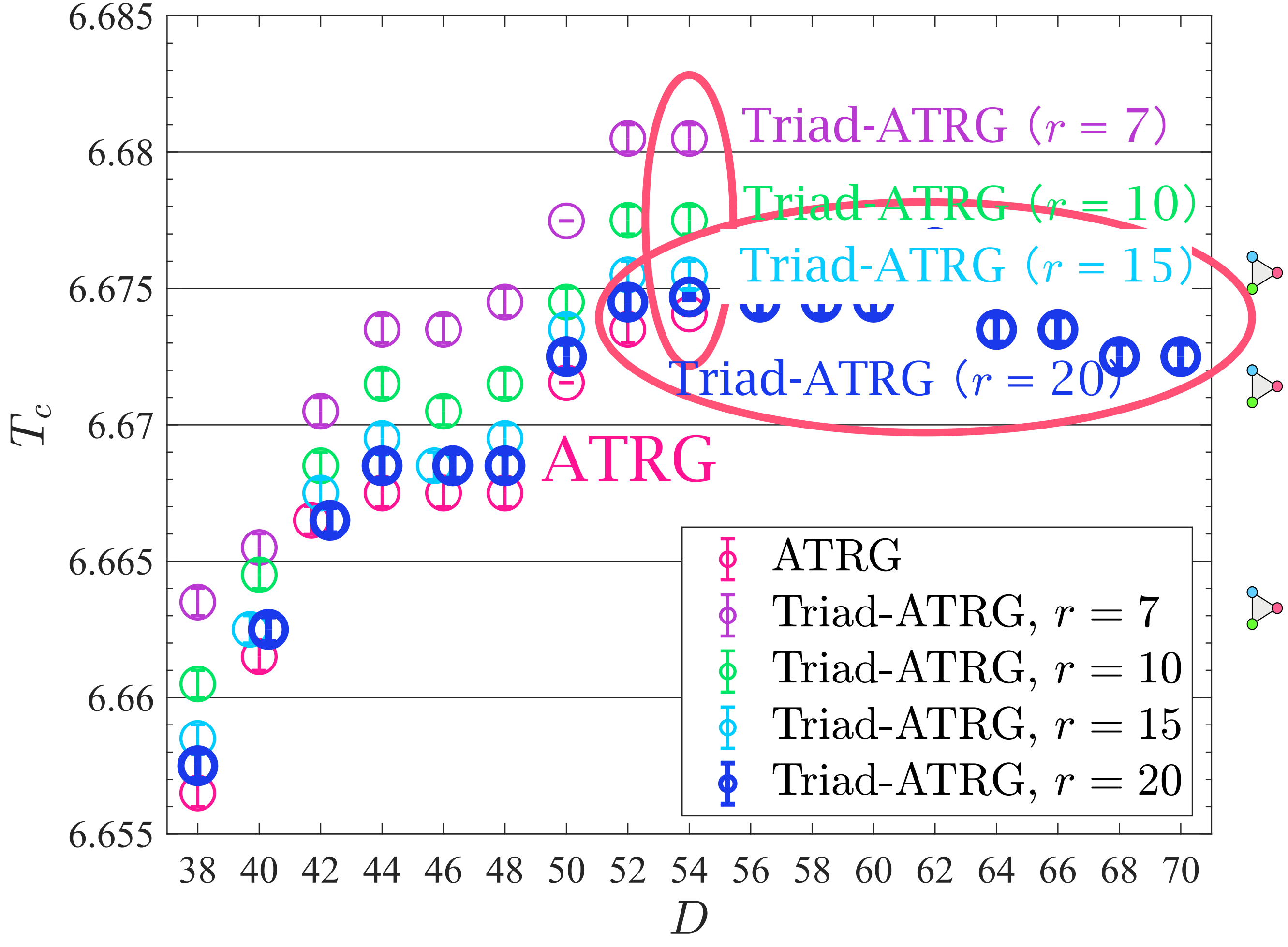
1.[S. Akiyama, Y. Kuramashi, and Y. Yoshimura, (2021).]



- compare ATRG and Triad-ATRG
- At $D = 54$, the error from ATRG is
  - 0.0013% ($r = 7$)
  - 0.00015% ($r = 20$)
- Extended Triad-ATRG up to $D = 70$
  (World record: $D = 55$, ATRG)[1]

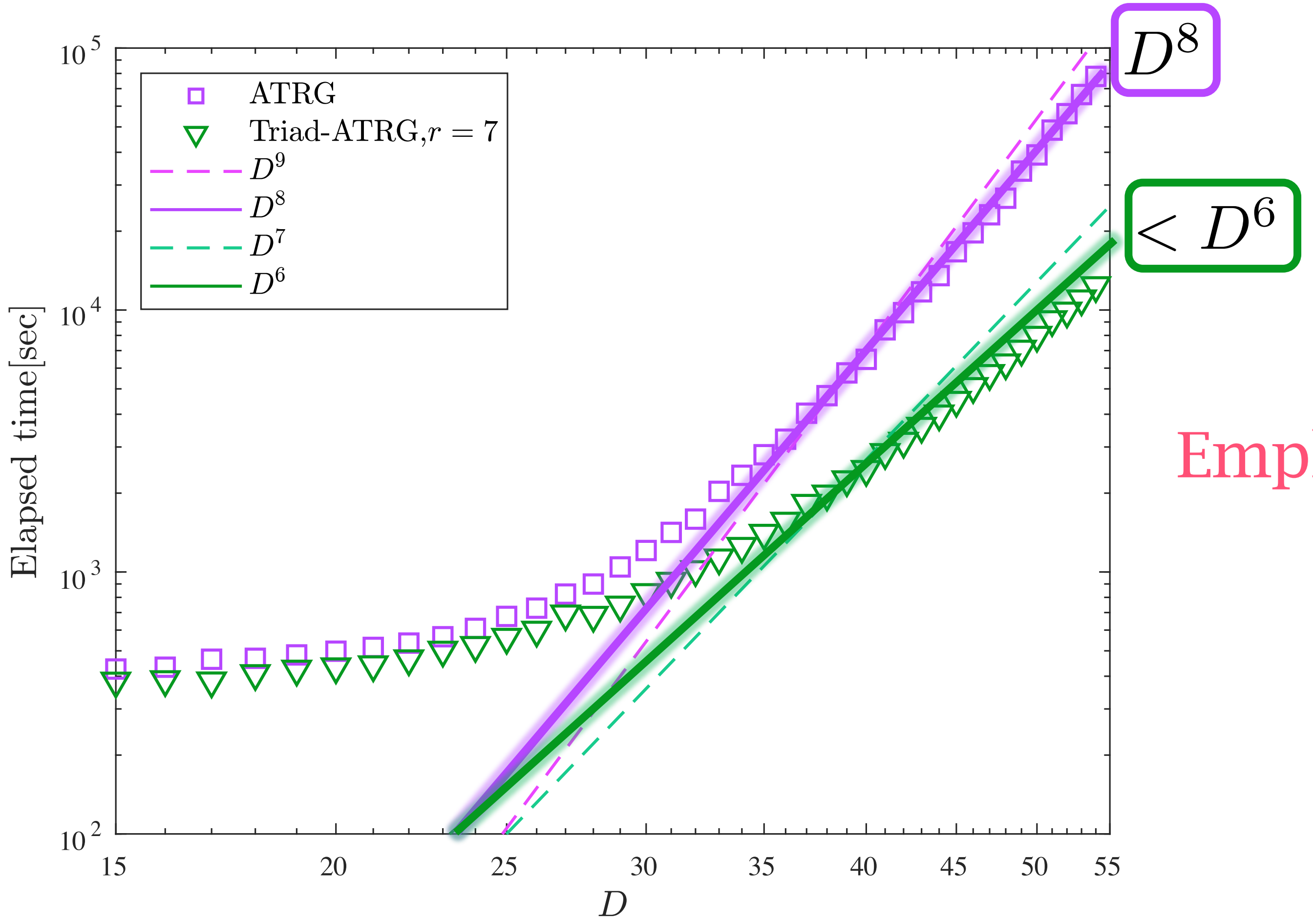
Triad-ATRG is comparable to ATRG

TRANSITION POINT



- Compare phase transition point
- at $D = 54$, difference is $0.09\%(r = 20)$
  →Transition point is also consistent
- extended Triad-ATRG for $D \geq 54$
  → Convergence is good

GPU PARALLELIZATION



- Practical cost reduction
- TRG is suitable for GPU

Employ TRG on GPUs (Julia language)

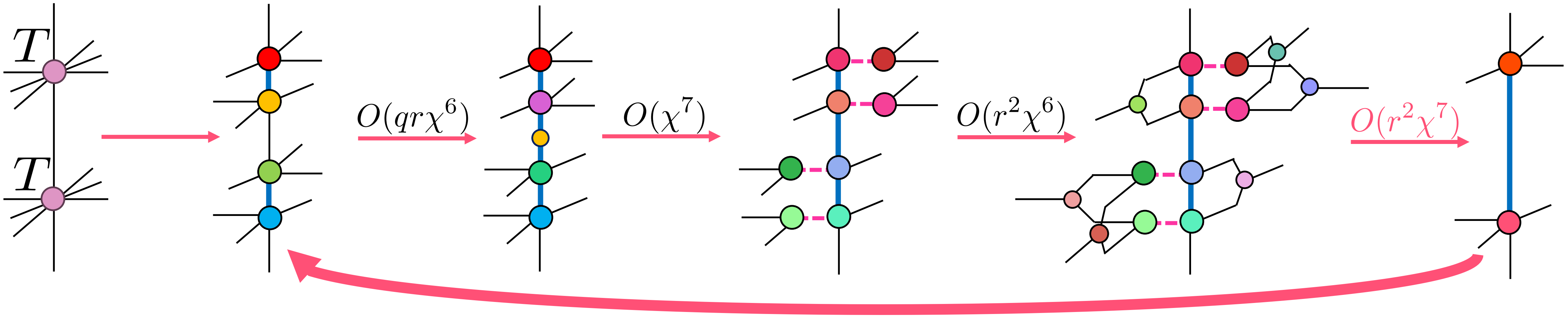
↓

- ATRG : $O(D^8)$
- Triad-ATRG : $O(D^6)$

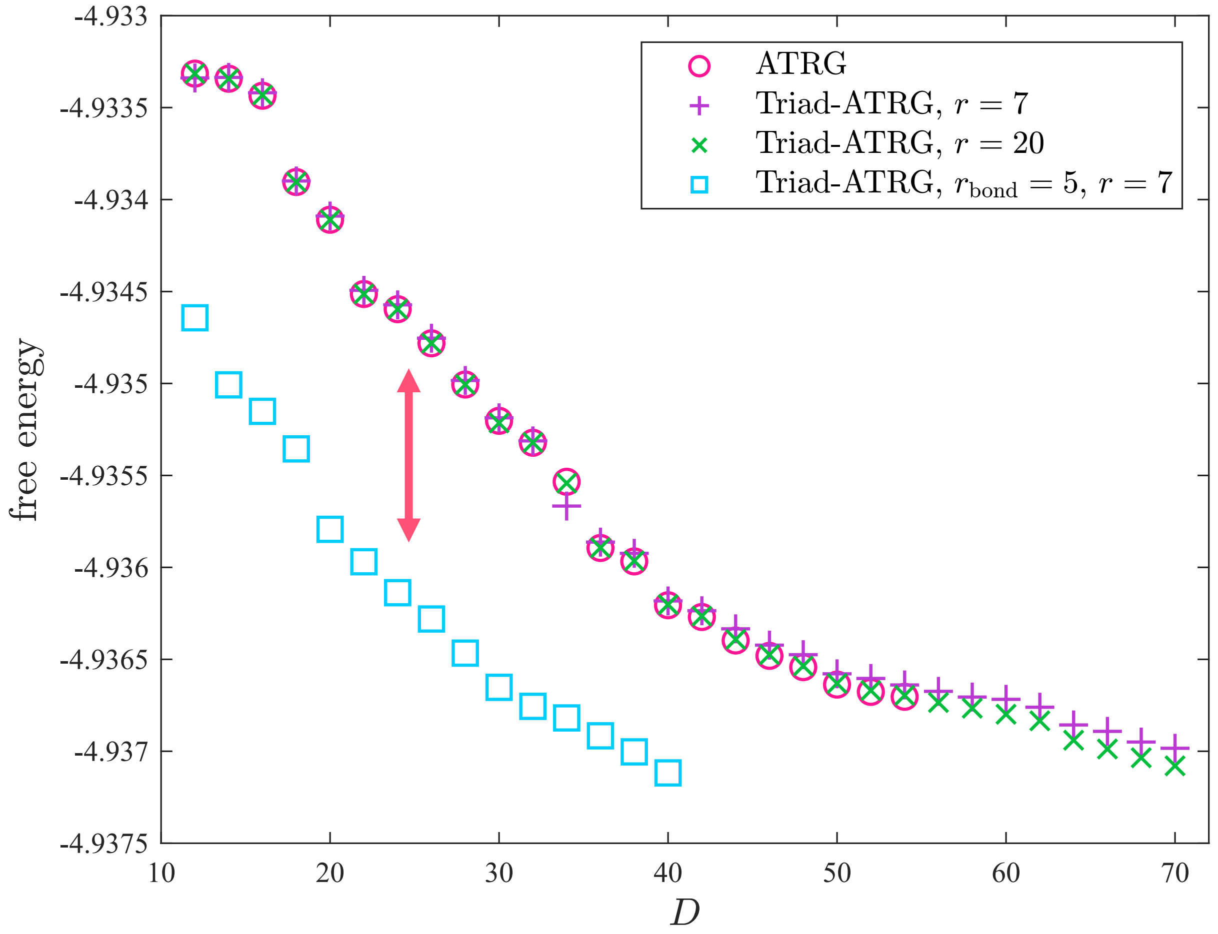
GPU computation reduces the cost ! Suitable for Triad-ATRG

OVERSAMPLING OF BOND-SWAPPING

- Consider oversampling of bond-swapping step
- Oversample r_{bond} times (blue line)
- ATRG: $O(r_{\mathrm{bond}}^2 D^9)$ 😰
- Triad-ATRG: $O(r_{\mathrm{bond}}^4 D^7 + r^2 r_{\mathrm{bond}}^2 D^7)$ 🙂 → ILO Triad-ATRG



OVERSAMPLING OF BOND-SWAPPING



- Oversampling of bond-swapping
- Lower free energy than ATRG

Problem

- Bond-swapping is bottleneck

→ Parallel implementation is needed

UNIVERSALITY CLASS OF 4D ISING MODEL

1. [M. Aizenman and R. Fernández, (1986).]
2. [F. J. Wegner and E. K. Riedel, (1973).]

- Analyze 4D Ising model using Triad-ATRG
- Critical exponents exactly follow mean-field values.[1]
- Perturbative analysis predicts logarithmic corrections[2]

$$m(T) \sim |T_c - T|^{\beta} |\ln|T_c - T||^{\tilde{\beta}}$$

$$m(h) \sim |h|^{\frac{1}{\delta}} |\ln|h||^{\tilde{\delta}}$$

m : magnetization
h : external field
T : temperature

- log corrections have been rigorously proven in ϕ^4 theory
- Monte Carlo (FSS) shows no clear evidence yet

SCALING COLLAPSE

- Assume mean-field critical exponents and test for logarithmic corrections
- Magnetization is expected to follow the form:

$$\frac{m}{h^{1-\frac{\gamma}{\Delta}}|\ln|h||^{\tilde{\gamma}+\frac{\gamma\tilde{\Delta}}{\Delta}}} = f\left(\frac{\tau^{\Delta}|\ln|\tau||^{\tilde{\Delta}}}{h}\right)$$

f : scaling function

γ, Δ : critical exponent

$\tilde{\gamma}, \tilde{\Delta}$: logarithmic corrections

$\tau = \frac{T-T_c}{T_c}$

[R. Kenna, D. A. Johnston, and W. Janke, (2006).]

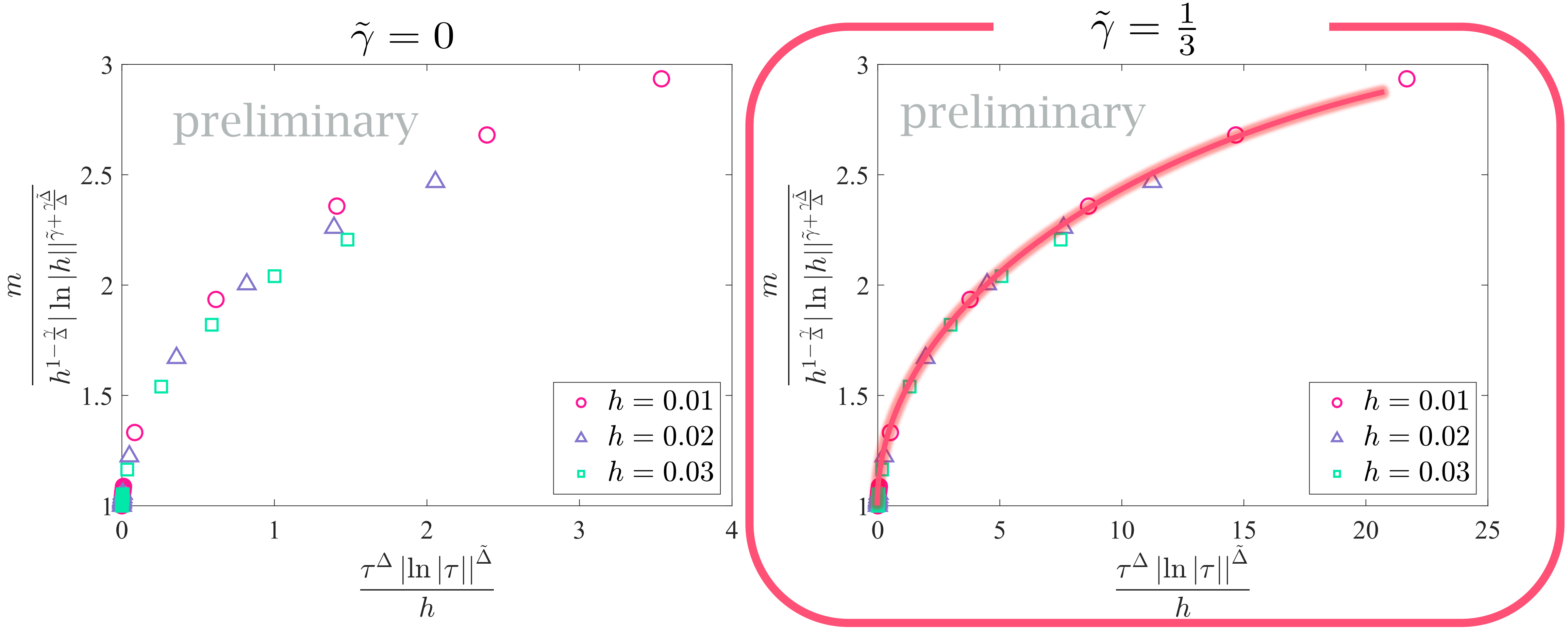
- Critical exponents : $\Delta = \frac{3}{2}, \gamma = 1$ (MF), $\tilde{\Delta} = 0, \tilde{\gamma} = \frac{1}{3}$ (ϕ^4 universality)
- Check scaling collapse for $m(\tau, h)$
- Investigation with Triad-ATRG ($D = 50, r = 20$, $V = 1024^4$)

cf. Monte-Carlo : $V = 256^4$

[P. H. Lundow and K. Markström, (2023).]

SCALING COLLAPSE

- plot with($\tilde{\gamma} = 0$) /without($\tilde{\gamma} = \frac{1}{3}$) logarithmic corrections

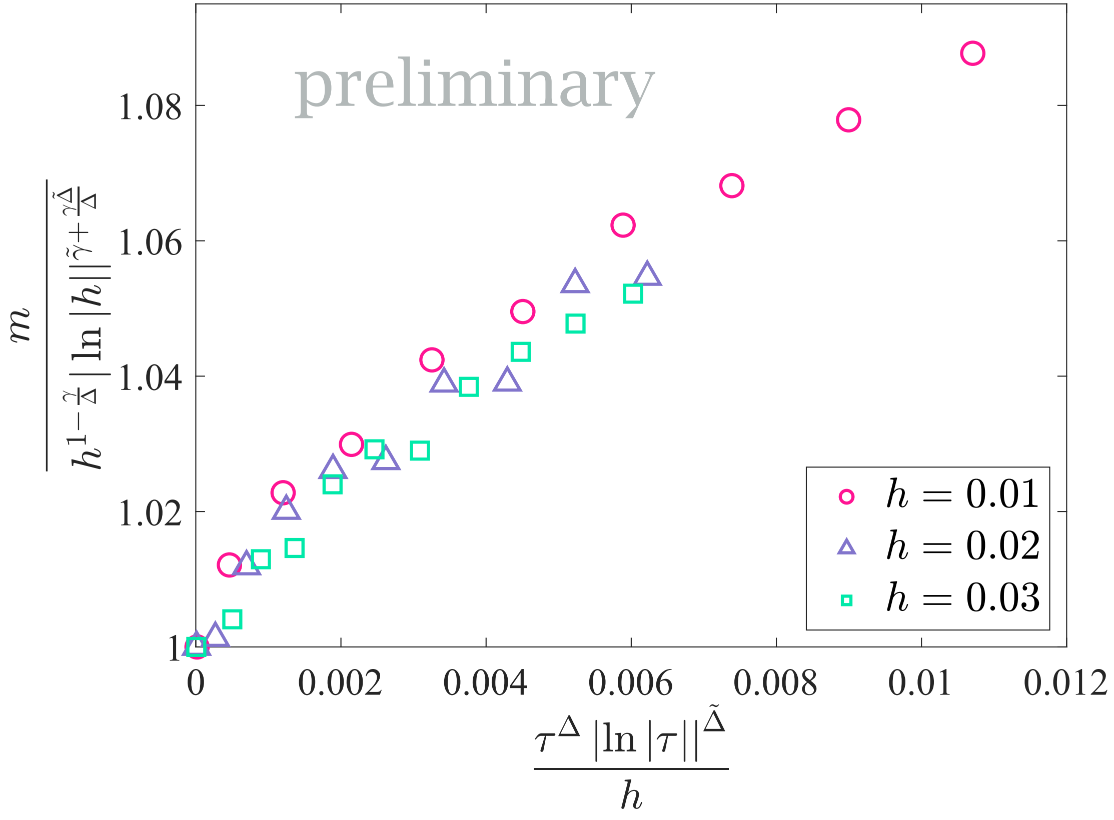


collapse with logarithmic corrections

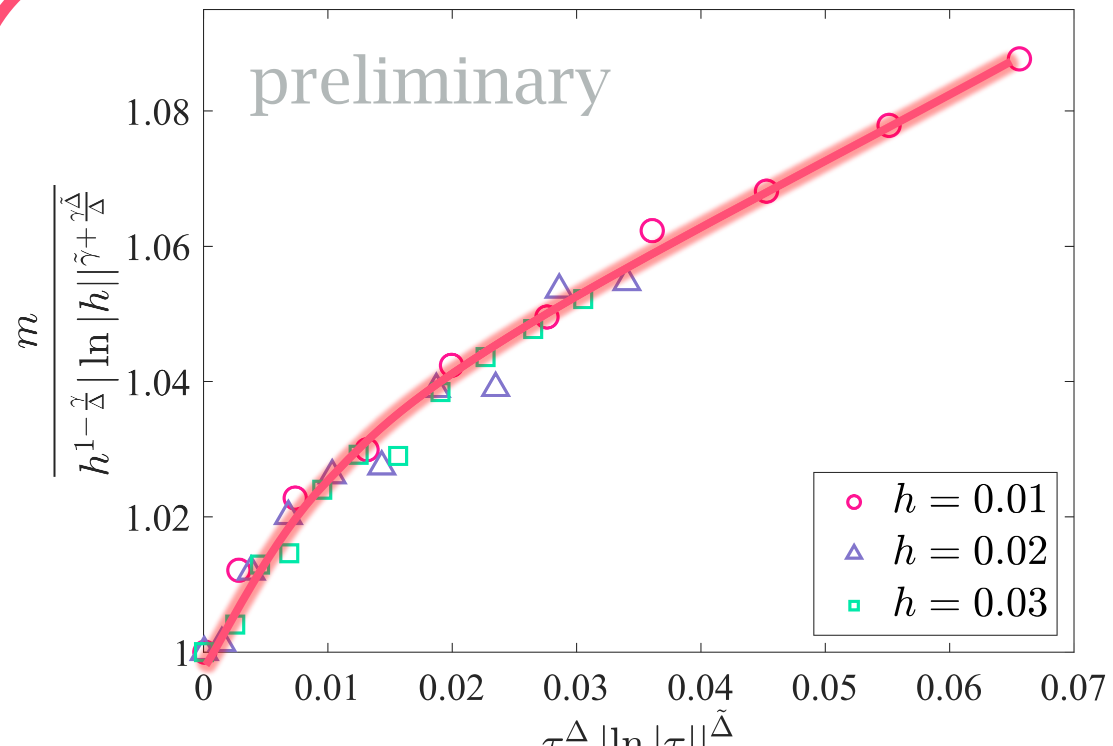
SCALING COLLAPSE

- plot with($\tilde{\gamma} = 0$) /without($\tilde{\gamma} = \frac{1}{3}$) logarithmic corrections

$\tilde{\gamma} = 0$



$\tilde{\gamma} = \frac{1}{3}$



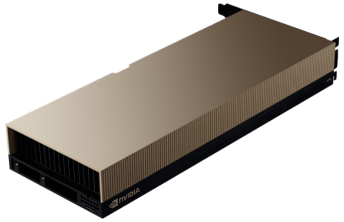
Indicating log corrections

SUMMARY

- Triad-ATRG achieved significant cost reduction from the ATRG
- Free energy are highly consistent with the ATRG
- Other quantities are also comparable }skipped
- Enables calculations in larger bond dimensions
- Triad-ATRG achieved higher prediction than the ATRG
- Application for critical exponent of Ising model

FUTURE WORK

www.elsa-jp.co.jp/wp-content/uploads/2023/01/85af85f357acd5e36904af6b3fe2f88a.png



- Enlarge bond dimensions (using NVIDIA H100?)
- Application for Grassmann tensor network and other system
- Estimation of critical exponents

Thank you for listening

END