Numerical Analysis of Quantum Entanglement Using Tensor Networks

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in collaboration with

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Tensor Network 2025

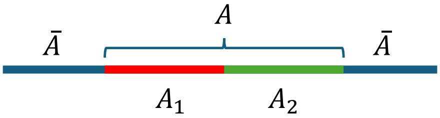
Quantum entanglement

- The non-local correlation between subsystems of a quantum many-body system.
- Wide applications in various topics:
  - Critical phenomena
  - Geometry of spacetime
  - Quantum circuits
- Various measures have been proposed:
  entanglement entropy, Rényi entropy, mutual information, etc.

Entanglement negativity $\mathcal{E}$ [Vidal-Werner, 2002]

- Measure of the quantum entanglement between two subsystems A_1, A_2.
  $A = A_1 \cup A_2$ can be in a mixed state!

$$\mathcal{E} \equiv \log \|\rho_A^{T_2}\| = \log \mathrm{tr}|\rho_A^{T_2}|$$



$\|M\|$: Sum of absolute values of eigenvalues of the matrix M (trace norm).

$\rho_A = \mathrm{tr}_{\bar{A}}\rho$: The reduced density matrix of the subsystem $A = A_1 \cup A_2$.

T_2: The partial transpose with respect to subsystem A_2.

$$(\rho_A)_{(I_1 I_2)(J_1 J_2)} = (\rho_A^{T_2})_{(I_1 J_2)(J_1 I_2)}$$

Replica trick

- Like the entanglement entropy, the entanglement negativity can be computed using the replica trick:

$$\mathcal{E} = \lim_{n_e \to 1} \log \mathrm{tr} \left(\rho_A^{T_2}\right)^{n_e} ,$$

where $n_e = 2, 4, 6, \ldots$.

Why entanglement negativity?

- EN allows us to probe quantum correlations in mixed states. (Mutual information cannot exclusively capture quantum correlations.)
- EN can be a useful probe for analyzing phase transitions in finite-temperature systems, such as QGP phase transition.
- We take advantage of the tensor renormalization group (TRG) approach to compute EN.

TRG approach [Levin-Nave, 2006]

- Coarse-graining of a given tensor network based on the idea of the real-space renormalization group.
    - Sign problem free.
    - Direct evaluation of the path integral (=density matrix).
    - No replica trick required.
- Applicable to higher-dimensional systems.

  (Computationally expensive for higher dimensions, but still feasible.)

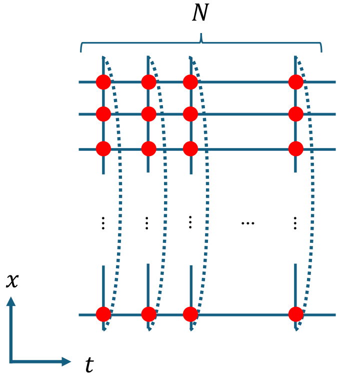
Tensor network representation of ρ

For example, We consider the density matrix $\rho = e^{-\beta H}/Z$ of a Gibbs state.

- For small $\Delta\beta$, $e^{-\Delta\beta H} \sim 1 - \Delta\beta H$ is a local operator acting on each lattice site.
- Tensors are locally connected to each other since the Hamiltonian H is local.
- Periodic boundary condition is imposed.

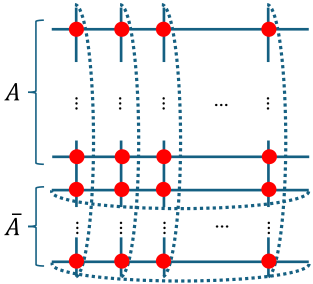
Tensor network representation of ρ

- Since $e^{-\beta H} = \lim_{N\to\infty} e^{-\beta/NH}$, $e^{-\beta H}$ is a product of local operators $e^{-\Delta\beta H}$, where $\Delta\beta = \beta/N$.
- If we consider the d-dimensional quantum system, ρ is represented as a $(d+1)$-dimensional tensor network.



Tensor network representation of ρ_A

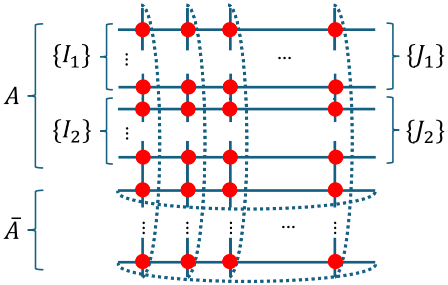
- The reduced density matrix ρ_A is obtained by tracing out the indices corresponding to the complementary subsystem $\bar{A}$.



Partial transpose $\rho_A^{T_2}$

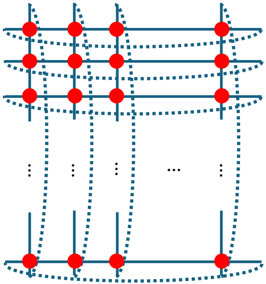
- We can swap the indices of the subsystem A_2 in ρ_A to obtain the partial transpose $\rho_A^{T_2}$.

$$(\rho_A^{T_2})_{(I_1 {\color{red}J_2})(J_1 {\color{red}I_2})} = (\rho_A)_{(I_1 {\color{red}I_2})(J_1 {\color{red}J_2})}$$



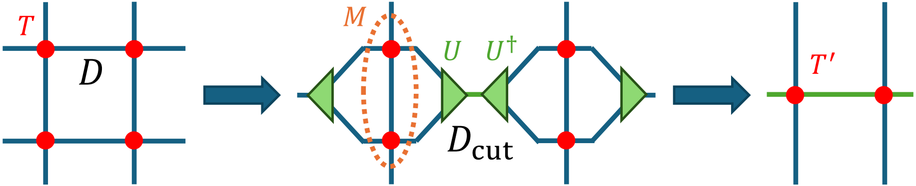
Tensor network representation of the partition function

- Of course, tracing out all the indices of ρ gives the partition function $Z = \mathrm{tr}\rho$.



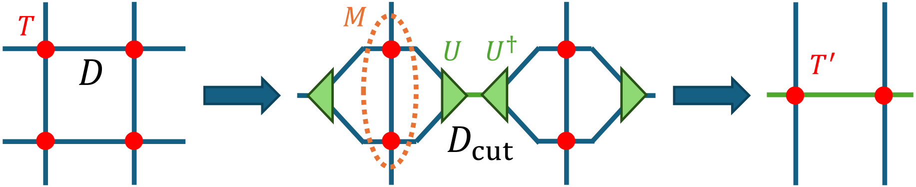
Higher-Order TRG algorithm [Xie et al., 2012]

We use the HOTRG algorithm for coarse-graining tensor networks.

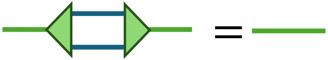


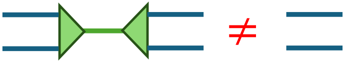
- Isometry matrix U diagonalizes the matrix $MM^\dagger$, and contains the D_{cut} largest eigenvalues of $MM^\dagger$.

HOTRG algorithm [Xie et al., 2012]



Note that

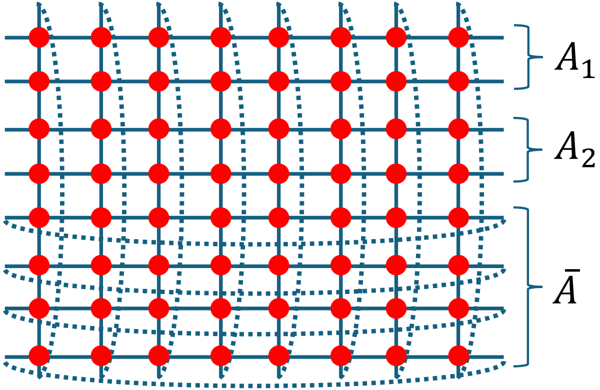




since the isometry matrix U is a submatrix of a unitary matrix.

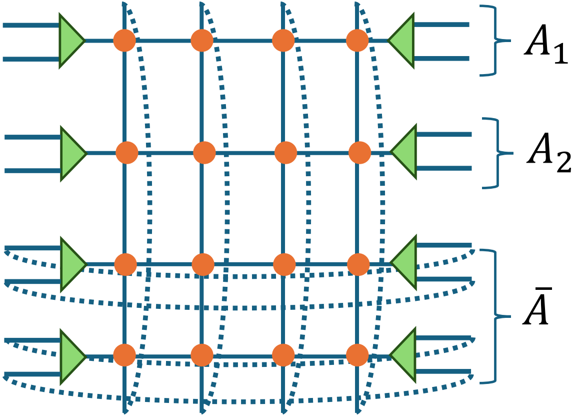
Coarse-graining of $\rho_A^{T_2}$

For example, we consider the reduced density matrix ρ_A shown below.



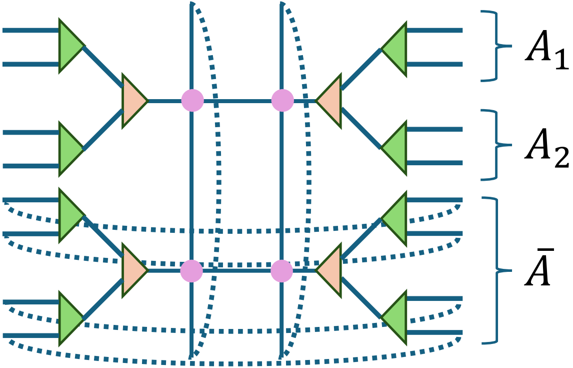
First set of coarse-graining

Apply the HOTRG algorithm once in both time and space directions.



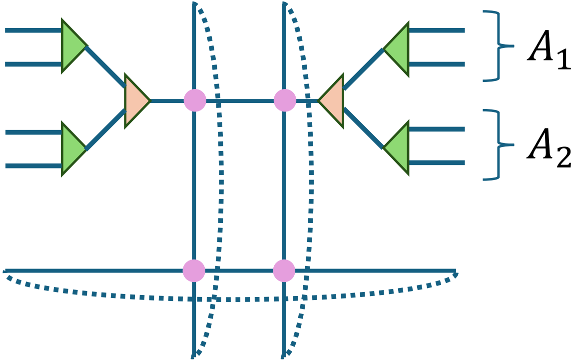
Second set of coarse-graining

Apply the HOTRG algorithm twice in both time and space directions.



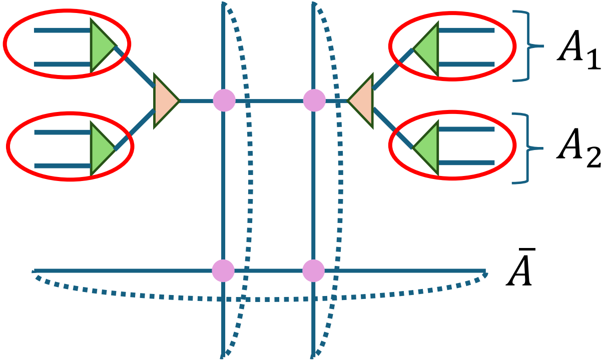
Further simplification

Using $U^\dagger U = 1$, we can simplify the lower part of the network.



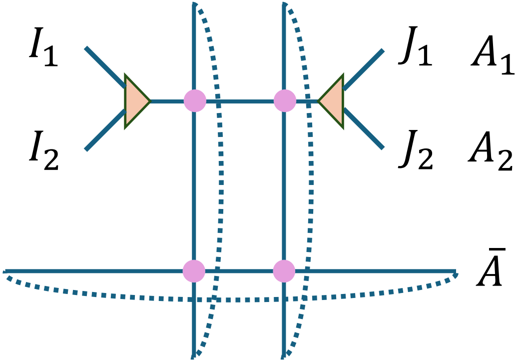
"Trimming" the network

Isometry matrices with two open indices do not contribute to the final result.



Final form of the network

Two sets of renormalization + simplification + trimming gives us the following network:



This network gives $(\rho_A^{T_2})_{(I_1 J_2)(J_1 I_2)}$.

Numerical test

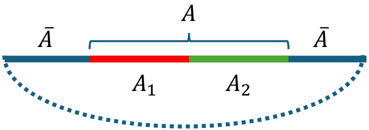
- We consider the one-dimensional quantum Ising model

$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h_x \sum_i \sigma_i^x ,$$

  at criticality $h_x = J$ and $T = 0$.
- We map the 1d quantum Ising model to the (1+1)d classical isometric Ising model at $T_{\mathsf{cl}} = T_{\mathsf{c}}$.

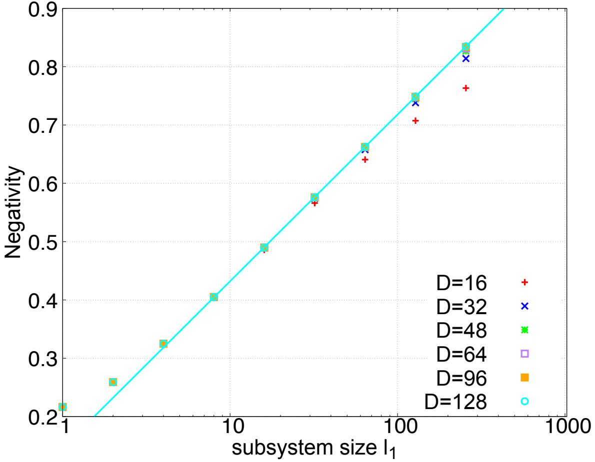
Numerical test

- We focus on the entanglement between two adjacent intervals A_1, A_2 in a larger system of size L with periodic boundary conditions.



- We fix the size of the subsystem such that $\ell_1 = \ell_2 = 1/4L$, and compute the entanglement negativity for various ℓ_1.

Entanglement negativity at criticality



- Analytic result:

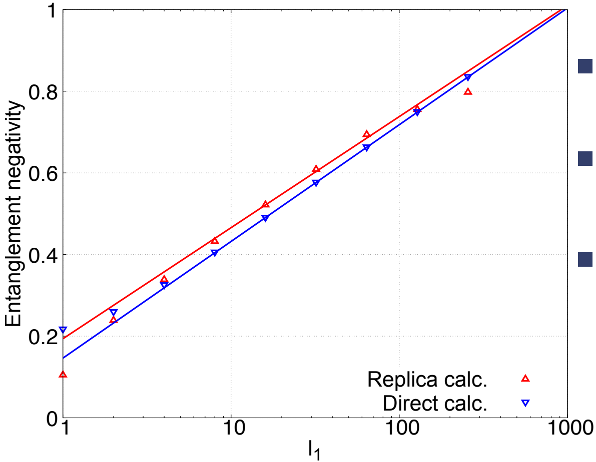
  [Calabrese et al., 2013]

$$\mathcal{E} = \frac{c}{4} \log \left(\frac{\ell_1}{\pi} \right) + k.$$

- Fitting result for $8 \leq \ell_1 \leq 128$: $c = 0.4966(9)$

  Theoretical value $c = 0.5$ is reproduced.

Replica vs direct calculation at $D_{\text{cut}} = 128$



- We extrapolate $\mathcal{E}$ to $n_e = 1$ for each $\ell_1 = 1, 2, 4, \ldots$.
- Fitting result for $8 \leq \ell_1 \leq 128$:

$$c = 0.47(2)$$

- cf. direct calculation results:

$$c = 0.44(1) \ (D_{\text{cut}} = 16)$$
$$c = 0.482(3) \ (D_{\text{cut}} = 32)$$

Conclusion and future prospects

Results

- We have computed the entanglement negativity of the one-dimensional quantum Ising model at criticality with TRG approach.
- Our result is consistent with known results, confirming the validity of our calculations.
- Our results implies the potential advantage of the TRG approach for computing entanglement negativity over the Monte Carlo method.

Conclusion and future prospects

Future prospects

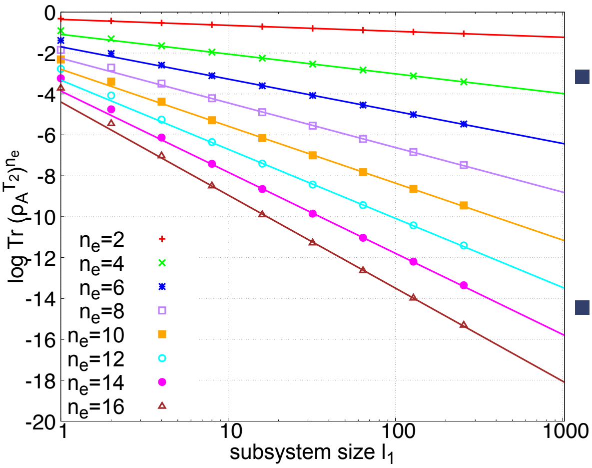
- More general configurations of subsystems

  General size of subsystems A_1, A_2, entanglement between disjoint intervals, higher dimensions, etc.

- Finite temperature systems

- Applications to quantum field theories

Backup slide: Details of the replica trick

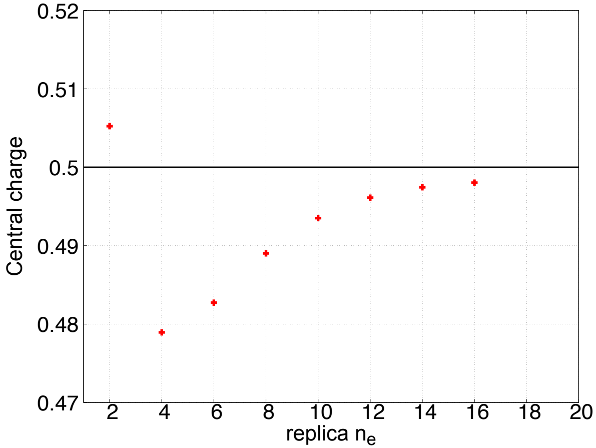


- Analytic result:
  [Calabrese et al., 2013]

$$\log \mathrm{Tr}(\rho_A^{T_2})^{n_e}$$
$$\simeq \log\left[\left(\frac{L^2}{2\pi^2}\right)^{-\frac{c}{6}(n_e/2-2/n_e)}\left(\frac{L}{2\pi}\right)^{-\frac{c}{6}(n_e/2+1/n_e)}\right]$$

- Analytic result exhibits special behavior at $n_e = 2$.

Backup slide: Details of the replica trick



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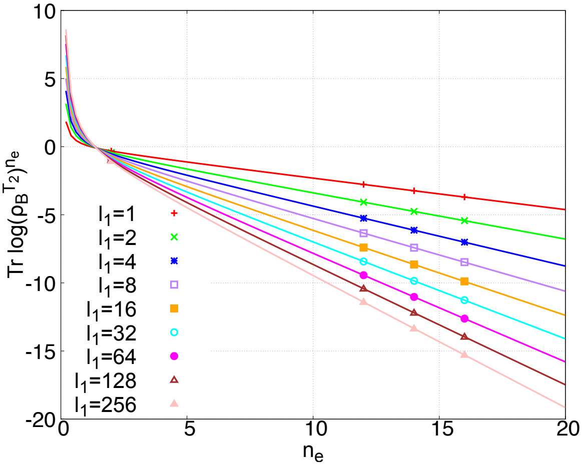
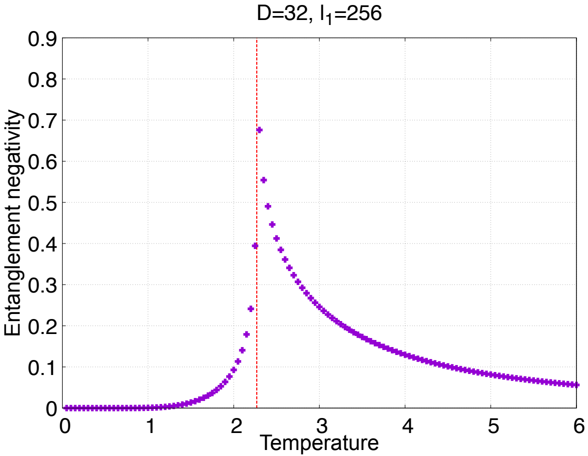


Backup slide: Temperature dependence of the entanglement negativity



Backup slide: Temperature dependence of the entanglement entropy