Efficient tensor network ansatz for abelian lattice gauge theory in (2+1)D

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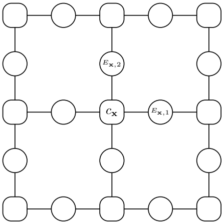
$U(1)$ lattice gauge theory: Hilbert space



- Gauge field on links: (in the electric basis)

$$E\,|n\rangle = n\,|n\rangle\,, \qquad n = 0, 1, 2, \cdots$$

$$[\phi, E] = i, \qquad U = e^{i\phi}, U\,|n\rangle = |n+1\rangle\,, \qquad \phi \sim \int_a^b \mathbf{A}\cdot d\mathbf{l}$$

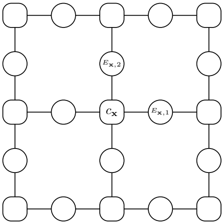
- Matter field on vertices: ($c_{\mathbf{x}}$: a boson or fermion operator on vertex of $\mathbf{x}$)

$$[c_{\mathbf{x}}, c_{\mathbf{y}}^{\dagger}]_{\pm} = \delta_{\mathbf{x},\mathbf{y}}$$

- Gauss's law at each vertex

$$c_{\mathbf{x}}^{\dagger}c_{\mathbf{x}} + E_{(\mathbf{x}-\mathbf{e}_1,1)} + E_{(\mathbf{x}-\mathbf{e}_2,2)} - E_{(\mathbf{x},1)} - E_{(\mathbf{x},2)} = q(\mathbf{x}), \qquad q(\mathbf{x}) \text{ pre-defined}$$

$U(1)$ lattice gauge theory: Hamiltonian



$$H = H_E + H_B + H_M$$

- Electric energy:

$$H_E = g\sum_{\mathbf{x},i} E^2_{\mathbf{x},i}, \qquad g > 0$$

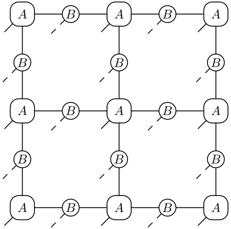
- Magnetic energy:

$$H_B = -h\sum_{\mathbf{x}} U_{\mathbf{x},1}U_{\mathbf{x}+\mathbf{e}_1,2}U^\dagger_{\mathbf{x}+\mathbf{e}_2,1}U^\dagger_{\mathbf{x},2} + h.c \sim -2h\cos(\oint_\square \mathbf{A}d\mathbf{l}), \qquad h > 0$$

- Matter energy:

$$H_M = \sum_{\mathbf{x}} m_{\mathbf{x}} c^\dagger_{\mathbf{x}} c_{\mathbf{x}} + J\sum_{\mathbf{x},i=1,2} c^\dagger_{\mathbf{x}} U_{\mathbf{x},i} c_{\mathbf{x}+\mathbf{e}_i} + h.c.$$

Tensor network representation of Z_N LGT states



Tensor network state with manifest gauge constraint:

- charge structure on each tensor leg

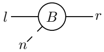
$$\text{tensor leg index: } i = 1, 2, \cdots, D$$

$$\text{charge function: } q(i) \to \{0, 1, 2, \cdots, N-1\}$$

$$\boxed{\text{Gauss's law: } c^\dagger_{\mathbf{x}} c_{\mathbf{x}} + E_{(\mathbf{x}-\mathbf{e}_1,1)} + E_{(\mathbf{x}-\mathbf{e}_2,2)} - E_{(\mathbf{x},1)} - E_{(\mathbf{x},2)} = q(\mathbf{x})}$$

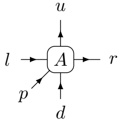
- gauge tensor B:

$$B^n_{lr} = B^n_{lr} \delta_{n,q(l)} \delta_{n,q(r)}$$



- matter tensor A:

$$A^p_{lrdu} = A^p_{lrdu} \delta_{q(s)+q(l)+q(d)-q(r)-q(u),q(\mathbf{x})}$$



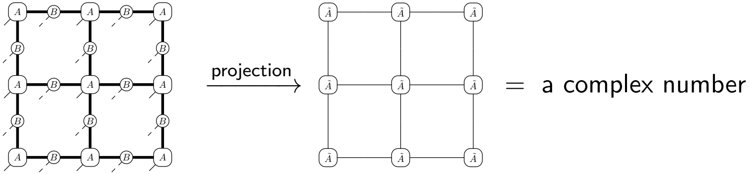
Optimization of gauge-invariant tensor network states

- Choose tensor B as the identity tensor *without losing representibility*:

$$B^n_{lr} = \delta_{lr}\delta_{n,q(l)}$$

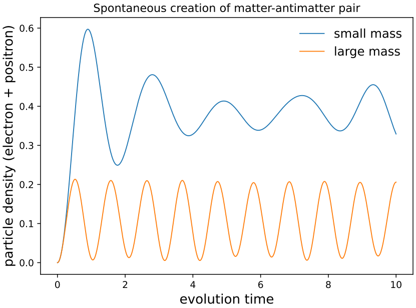
  B can be frozen in the optimization of the tensor network states.

- One way to find the ground states of the LGT is to do variational Monte Carlo. In VMC, the key quantity to evaluate is the amplitude of the projection of the wavefunction onto a gauge field configuration **n** and a matter field configuration **p**. Due to the above structure of B, evaluation of the projected amplitude is equivalent to the contraction of a 2D conventional TNS with **reduced** bond dimension:



QED in (1+1)D: particle creation in real time

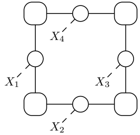
$$H = m\sum_x (-1)^x c_x^\dagger c_x + J\sum_x (c_x^\dagger U_{x+\frac{1}{2}} c_{x+1} + h.c.) + g\sum_x E^2_{x+\frac{1}{2}}, \qquad U \equiv e^{i\phi},\ [\phi, E] = i$$



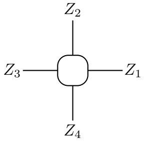
small mass $\rightarrow$ more particles created
large mass $\rightarrow$ less particles created

Easy numerics, but deep physics.

Even Z_2 gauge theory



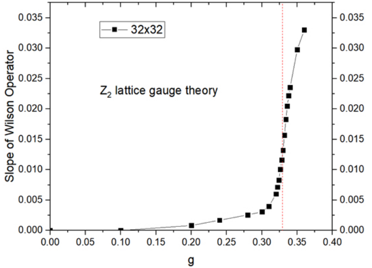
$$H = -g \sum_l Z_l - \sum_\square X_1 X_2 X_3 X_4,$$ on each vertex $Z_1 Z_2 Z_3 Z_4 = 1$



Ground state optimization: variational Monte Carlo

- efficient: $\sim O((\frac{D}{N})^7)$ for Z_N gauge theory
- accurate: $e_{\mathrm{PEPS}} = -0.69092(6)$ vs $e_{\mathrm{QMC}} = -0.69083(6)$ for 16×16, $g \approx g_c$.

Fit $\log(W)$ vs area$(\mathcal{C})$

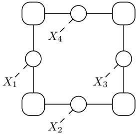


Wilson loop (over a square loop $\mathcal{C}$):

$$W = \prod_{l \in \mathcal{C}} X_l \sim \begin{cases} e^{-k \text{ Area}(\mathcal{C})} & g > g_c \\ e^{-k' \text{ Perimeter}(\mathcal{C})} & g < g_c \end{cases}$$

g_c known from duality to 2D transverse field Ising model.

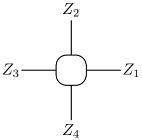
Odd Z_2 gauge theory



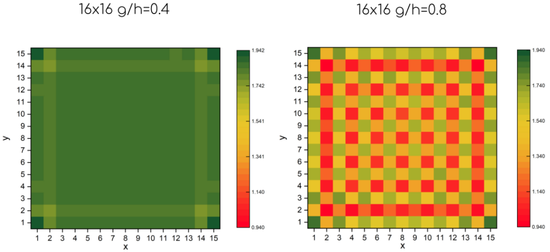
Z_2 odd gauge theory in (2+1)D:

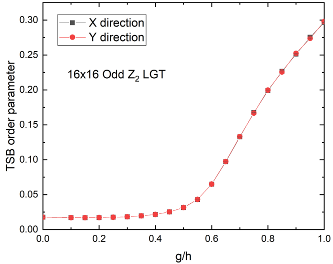
$$H = -g \sum_{l} Z_l - h \sum_{\square} X_1 X_2 X_3 X_4,$$

on each vertex $Z_1 Z_2 Z_3 Z_4 = -1$



Corresponds to a frustrated spin system.





Distribution of $(X_1 X_2 X_3 X_4)_{\mathbf{i}}$ at ground state.

Translation symmetry breaking (TSB) for large g.

$$\text{TSB OP} \equiv D_x = \frac{1}{N} \sum_{\mathbf{i}} (-1)^{i_x} (X_1 X_2 X_3 X_4)_{\mathbf{i}}$$

Future

Still a long way to go to reach QCD

- $Z_N \to SU(3)$
- $(2+1)D \to (3+1)D$
- no matter $\to$ fermionic matter (straightforward)

Discretize space by lattice

Other directions:

- full update, simple update
- infinite PEPS
- continuous space MPS
- SPT and generalized symmetry in lattice gauge theory
  - e.g. Higgs phase $\leftrightarrow$ SPT: boundary phase transitions in Z_2 LGT with matter fields
  - ...

Special thanks to my collaborator

Professor Wenyuan Liu, Zhejiang University